

AD-A158 194

THE H-P VERSION OF THE FINITE ELEMENT METHOD(U)  
MARYLAND UNIV COLLEGE PARK LAB FOR NUMERICAL ANALYSIS  
I BABUSKA ET AL. JUL 85 BN-1043 N00014-85-K-0169

1/1

UNCLASSIFIED

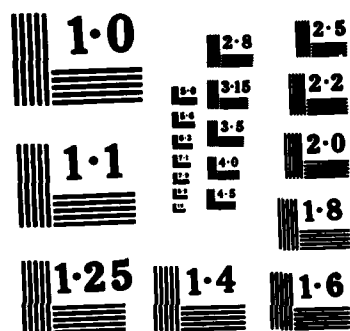
F/G 12/1

NL

END

FILED

...



NATIONAL BUREAU OF STANDARDS  
MICROCOPY RESOLUTION TEST CHART



INSTITUTE FOR PHYSICAL SCIENCE  
AND TECHNOLOGY

2

Laboratory of Numerical Analysis *1/2*

Technical Note BN-1043

AD-A158 194

THE h-p VERSION OF THE FINITE ELEMENT METHOD

by

B. Guo and I. Babuška

DTIC FILE COPY

DTIC  
ELECTE  
AUG 21 1985  
S D  
E

July 1985

This document has been approved  
for public release and sale; its  
distribution is unlimited.



UNIVERSITY OF MARYLAND

85

8

20

002

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1043	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The h-p Version of the Finite Element Method		5. TYPE OF REPORT & PERIOD COVERED Final life of the contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) B. Guo and I. Babuska		8. CONTRACT OR GRANT NUMBER(s) ONR-N00014-85-K-0169
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE July 1985
		13. NUMBER OF PAGES 84
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release: distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The paper analyzes the h-p version of the finite element method in two dimensions. It proves that the error of the method measured in the energy norm decreases exponentially. Extensive computational analysis with the code PROBE, shows the practical effectivity of the h-p version.		

The h-p Version of the Finite Element Method

B. Guo\*

I. Babuška\*\*

\*Supported by the NSF Grant DMS-8315216

\*\*Partially supported by ONR Contract N00014-85-K-0169

# ABSTRACT

The paper analyzes the h-p version of the finite element method in two dimensions. It proves that the error  $\|e\|_E$  of the method measured in the energy norm decreases exponentially:  $\|e\|_E \leq Ce^{-b\sqrt[3]{N}}$  where N is the number of degrees of freedom. The exact solution, which is approximated by the finite element method, is assumed to belong to the space  $B_{\beta}^{2,2}$ . This space contains the solutions of the problems of elliptic partial differential equations with piecewise analytic data, such as, when the domain has corners, the boundary conditions and the coefficients of the equations are piecewise analytic, etc. Extensive computational analysis with the code PROBE, shows the practical effectivity of the h-p version, and the applicability of the theoretical asymptotic error estimates in the range of engineering computations and accuracy.

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Avail and/or	
Dist	Special
A-1	



## 1. INTRODUCTION

The  $h$ ,  $p$  and  $h$ - $p$  versions are the three basic approaches of the finite element method when the finite dimensional subspace is composed of piecewise polynomials defined on a partition of a given domain. The convergence is achieved by increasing the dimension of the finite element subspace in three different ways.

The first one is called the  $h$ -version when the degree  $p$  of the polynomials is fixed (at a low value) and the mesh size  $h$  is reduced to obtain the desired accuracy. The  $h$ -version has been investigated theoretically and practically for many years. There are many computer programs of the  $h$ -version which are used in engineering. One of the popular codes is for example NASTRAN in its various versions, e.g. MCS/NASTRAN and others are ADINA, ANSYS, etc.

In the  $p$ -version the mesh is fixed and the degree  $p$  of the polynomials is increased in order to reduce the approximation error. The development of the  $p$ -version is very recent. PROBE is currently the only commercial code (released in 1985) using the  $p$  and  $h$ - $p$  versions.

The  $h$ - $p$  version combines the  $h$  and  $p$ -versions.

The  $p$ -version was first theoretically studied in [7]. The  $h$ - $p$  version was addressed in [3]. It was conjectured there that it is possible to achieve exponential rate convergence in the cases of practical importance. For additional features of the  $p$ -version we refer also to [6]. The  $p$ -version in three dimensions was analyzed in [12], [13] and for the detailed analysis of the  $p$  and  $h$ - $p$  version in one dimension see [16]. For the engineering and implementational aspect of the  $p$  and  $h$ - $p$  version we refer to [4].

It has been proven that the rate of convergence for the  $p$ -version cannot be worse than that of the  $h$ -version with a quasiuniform mesh [7]. If the singularity of the solution is located at vertices, then the rate of convergence of the  $p$ -version is at least twice that of the  $h$ -version with quasiuniform mesh. It will be shown in this paper that under proper assumptions satisfied usually in practice, the  $h$ - $p$  version has exponential rate of convergence with respect to the number of degrees of freedom, while the  $h$  and  $p$ -versions have only a polynomial rate [2].

The singular behaviour of the solution of partial differential equations of elliptic type is typically caused by piecewise smoothness of the input data, by the corners and edges of the domain, etc. Usually, in practice the data are piecewise analytic functions.

In Chapter 2 we introduce the spaces  $H_{\beta}^{k,\ell}(\Omega)$  ( $k \geq \ell \geq 0$ ) which are a generalization of the weighted Sobolev spaces used in [5]. The main tool of the analysis in the present paper is the countable normed space  $B_{\beta,d}^{\ell}(\Omega)$  which consists of all functions  $u$  belonging to  $H_{\beta}^{k,\ell}(\Omega)$  for all  $k$  and  $\|u\|_{H_{\beta}^{k,\ell}(\Omega)} \leq Cd^{k-\ell}(k-\ell)!$  ( $\ell=0,1,2$ ). It can be shown [17] that the solutions of partial differential equations of elliptic type with piecewise analytic data belong to this space.

Some imbedding inequalities related to the spaces  $H_{\beta}^{2,2}(\Omega)$  are derived in Chapter 3.

The accuracy of the finite element method reduces to an approximation problem when the coercivity or the "inf-sup" condition is satisfied [2] [11]. We will study the approximation (in the space  $H^1$ ) of functions  $u \in B_{\beta,d}^2(\Omega)$  by the  $h$ - $p$  version and will show that exponential rate of convergence with respect to the number of degrees of freedom can be achieved.

In Chapter 4 we analyze approximation properties of the polynomials on a single square and a parallelogram.



In Chapter 5 a geometric mesh on a square and a parallelogram is introduced and the exponential rate of the convergence of the h-p version is proven.

Chapter 6 generalizes the results of chapter 5 by introducing general geometric meshes composed of curvilinear quadrilaterals and triangles.

The last chapter addresses the numerical results and the performance of the h-p versions by the computational analysis of an elasticity problem by the code PROBE.

## 2. NOTATION AND PRELIMINARIES

We shall denote integers by  $i, j, k, \ell, m$  and  $n$ , and by  $\mathbb{R}^1$  and  $\mathbb{R}^2$  we shall denote the one and two dimensional Euclidean space with  $x = (x_1, x_2)$  or  $x = (x, y)$ . If  $Q$  is any one or two dimensional set,  $\bar{Q}$  denotes its closure. By  $\Omega$  we denote a polygonal domain in  $\mathbb{R}^2$  with the boundary  $\partial\Omega = \Gamma$ , the vertices  $A_i$ ,  $1 \leq i \leq M$  and  $\Gamma_i$ ,  $1 \leq i \leq M$  the open edge of  $\partial\Omega$  linking  $A_{i+1}$  and  $A_i$ , ( $A_0 = A_M$ ). We have  $\partial\Omega = \bigcup_{i=1}^M \bar{\Gamma}_i$ , where  $\bar{\Gamma}_i$  is the closure of  $\Gamma_i$ . The measure of the interior angle of  $\Omega$  at  $A_i$  is denoted by  $\omega_i$ .

By  $H^m(\Omega)$  (resp.  $H^m(\mathbb{R}^2)$ ) ( $m \geq 0$ ) we denote the Sobolev space of functions on  $\Omega$  (resp.  $\mathbb{R}^2$ ) with square integrable derivatives of order  $\leq m$  ( $m \geq 0$ ) furnished with the norm

$$\|u\|_{H^m(\Omega)}^2 = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^2(\Omega)}^2$$

where  $\alpha = (\alpha_1, \alpha_2)$ ,  $\alpha_i \geq 0$ ,  $i = 1, 2$ , integers,  $|\alpha| = \alpha_1 + \alpha_2$ , and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = u \frac{\alpha_1}{x_1} \frac{\alpha_2}{x_2}.$$

As usual  $H^0(\Omega) = L^2(\Omega)$ . Further we will use the notation

$$\|u\|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{H^0(\Omega)}^2$$

and

$$|D^m u|^2 = \sum_{|\alpha| = m} |D^\alpha u|^2.$$

By  $r_i(x)$  we shall denote the Euclidean distance between  $x$  and the vertex  $A_i$  of  $\Omega$ . Let  $\beta = (\beta_1, \beta_2, \dots, \beta_M)$  be an  $M$ -tuple of real numbers,

$0 < \beta_i < 1$ ,  $1 \leq i \leq M$ . For any integer  $k$  we shall write  
 $\beta \pm k = (\beta_1 \pm k, \beta_2 \pm k, \dots, \beta_M \pm k)$ . We denote  $\Phi_\beta(x) = \prod_{i=1}^M r_i^{\beta_i}(x)$  and  
 $\Phi_{\beta \pm k}(x) = \prod_{i=1}^M r_i^{\beta_i \pm k}(x)$ .

By  $H_\beta^{m,\ell}(\Omega)$ ,  $m \geq \ell \geq 0$  ( $H_\beta^{m,0}(\Omega) = H_\beta^m(\Omega)$ ) we denote the completion of the set of all infinitely differentiable functions under the norm

$$\|u\|_{H_\beta^{m,\ell}(\Omega)}^2 = \|u\|_{H_\beta^{\ell-1}(\Omega)}^2 + \sum_{k=\ell}^m \int_{\Omega} |D^k u(x)|^2 \Phi_{\beta+k-\ell}^2(x) dx \quad (\ell \geq 1)$$

and

$$\|u\|_{H_\beta^{m,0}(\Omega)}^2 = \|u\|_{H_\beta^0(\Omega)}^2 = \sum_{k=0}^m \int_{\Omega} |D^k u(x)|^2 \Phi_{\beta+k}^2(x) dx \quad (\ell = 0).$$

For  $m = \ell = 0$ ,  $H_\beta^{0,0}(\Omega) = L_\beta^2(\Omega)$ . Analogously as before

$$\|u\|_{H_\beta^{m,\ell}(\Omega)}^2 = \sum_{k=\ell}^m \int_{\Omega} |D^k u(x)|^2 \Phi_{\beta+k-\ell}^2(x) dx.$$

The space  $H_\beta^{m,2}(\Omega)$  was introduced in [3] where its various properties were studied. Let us mention the following lemma proven in [5] which will be needed later.

Lemma 2.1.  $H_\beta^{2,2}(\Omega) \subset C^0(\bar{\Omega})$  with continuous imbeddings i.e.,

$$\sup_{x \in \bar{\Omega}} |u| \leq C \|u\|_{H_\beta^{2,2}(\Omega)}$$

where  $C$  depends on  $\Omega$  but is independent of  $u$ . □

Let

$$(2.1) \quad \Psi_\beta^\ell(\Omega) = \{u(x) \mid u \in H_\beta^{m,\ell}(\Omega), \quad \forall m \geq \ell, \quad 0 \leq \ell \leq 2\}$$

and

$$(2.2) \quad B_{\beta}^{\ell}(\Omega) = \{u(x) \mid u \in \Psi_{\beta}^{\ell}(\Omega), (\int_{\Omega} |D^{\alpha} u|^2 \phi_{\beta+k-\ell}^2 dx)^{1/2} \leq C d^{(k-\ell)} (k-\ell)!\}$$

for  $|\alpha| = k = \ell, \ell+1, \dots, 0 \leq \ell \leq 2, d \geq 1$ , constants  $C$  and  $d$  independent of  $k$

be the countably normed spaces (see [15]). For  $\ell = 0$  we shall write  $B_{\beta}(\Omega)$  instead of  $B_{\beta}^0(\Omega)$ . The functions in  $B_{\beta}^{\ell}(\Omega)$  are characterized by different constants  $C$  and  $d$ . If we would like to emphasize the dependence on the constant  $d$  we will write  $B_{\beta,d}^0(\Omega) = B_{\beta,d}(\Omega)$ , etc.

The weighted Sobolev spaces  $H_{\beta}^{s,\ell}(\Omega)$  of a non-integral  $s$  are defined as the interpolation spaces. Let  $\mathcal{B}$  denote the category of all  $H_{\beta}^{k,\ell}(\Omega)$ ,  $k = \ell, \ell+1, \dots$ .  $\mathcal{B}$  is a sub-category of  $\mathcal{N}$ , which denotes the category of all Banach spaces,  $H_{\beta}^{k,\ell}(\Omega)$  ( $k \geq \ell \geq 0$ ) is a normed space, and  $H_{\beta}^{k+1,\ell}(\Omega) \subset H_{\beta}^{k,\ell}(\Omega)$ . Hence  $\bar{A} = (H_{\beta}^{k,\ell}(\Omega), H_{\beta}^{k+1,\ell}(\Omega))$  is compatible couple in  $\mathcal{B}$ . By the application of the  $K$ -method, we define the interpolation space (see [10]):

$$(H_{\beta}^{k,\ell}(\Omega), H_{\beta}^{k+1,\ell}(\Omega))_{\theta,\infty} = H_{\beta}^{k+\theta,\ell}(\Omega), \quad 0 < \theta < 1.$$

It is easy to verify that if  $u \in B_{\beta,d}^{\ell}(\Omega)$ , then for any  $k \geq \ell$

$$(2.3) \quad \|u\|_{H_{\beta}^{k+\theta,\ell}(\Omega)} \leq C d^{k+\theta-\ell} (k+\theta)^{1/2} \Gamma(k+1-\ell+\theta).$$

## 3. SOME IMBEDDING INEQUALITIES

We shall generalize some imbedding inequalities proven in [2] [3] .

$T$  be a rectangle (resp. triangle) with vertices  $(x_i, y_i)$ ,  $1 \leq i \leq 4$  (resp.  $1 \leq i \leq 3$ );  $(x_1, y_1)$  is supposed to be the origin. Let  $h$  and  $\underline{h}$  be the length of the longest and shortest side of  $T$ ,  $\omega_i$  be the measure of the angles of  $T$ ,  $1 \leq i \leq 4$  (resp. 3). We assume that there are constants  $\kappa$  and  $\omega_0 > 0$  such that for all rectangles (resp. triangles) under consideration

$$h/\underline{h} \leq \kappa < \infty$$

and

$$0 < \omega_0 \leq \omega_i \leq \pi - \omega_0 < \pi.$$

Let  $\gamma$  be the boundary or some sides of  $T$ ,  $H^k(\gamma) = \prod_{i=1}^M H^k(\gamma_i)$ ,  $k = 0, 1$  and  $M \leq 4$  (resp.  $M \leq 3$ ) where  $\gamma_i$ 's are the sides of  $T$ , and  $H^k(\gamma_i)$  is the Sobolev space on  $\gamma_i$ .

Lemma 3.1. Let  $u \in H^2(T)$ , and  $u$  vanishes at vertices of  $T$ , then  $u \in H^1(\gamma)$  and

$$(3.1) \quad \|u\|_{H^1(\gamma)}^2 \leq Ch \|u\|_{H^2(T)}^2.$$

Proof. Let  $S$  be the standard square  $(0,1) \times (0,1)$ . We obtain by the standard imbedding theorem (with  $C > 0$  independent of  $u$ ):

$$\|u\|_{L^2(\gamma)}^2 \leq C \|u\|_{H^1(S)}^2 \leq C (|u|_{H^1(S)}^2 + |u|_{L^2(S)}^2),$$

$$|u|_{H^1(\gamma)}^2 \leq C (|u|_{H^2(S)}^2 + |u|_{H^1(S)}^2).$$

Since  $u$  vanishes at vertices of  $S$ , there is some  $C_1 > 0$  independent of  $u$  (see [1],[2],[11]) such that

$$\|u\|_{L^2(S)} \leq C_1 |u|_{H^2(S)}$$

and

$$|u|_{H^1(S)} \leq C_1 |u|_{H^2(S)}.$$

Hence

$$\|u\|_{H^1(\gamma)}^2 \leq C_2 |u|_{H^2(S)}^2.$$

The usual scaling argument yields (3.1). The proof for the triangle is the same. □

Lemma 3.2. Let  $u \in H_{\beta}^{2,2}(T)$  with  $\Phi_{\beta} = r^{\beta}$ ,  $0 < \beta < 1$  and  $u$  vanishes at the vertices of  $T$ . If  $\gamma$  is a side of  $T$  separated from origin then

$$(3.2) \quad \|u\|_{H^1(\gamma)}^2 \leq h^{1-2\beta} \|u\|_{H^{2,2}(T)}^2.$$

Proof: Let  $S$  be the standard square  $(0,1) \times (0,1)$  and  $S_1 = (0,1) \times (1/2,1)$ .

Suppose  $\gamma$  is the top edge of  $S$ . We have by applying Lemma 3.1 (see also

Lemma 2.1):

$$\begin{aligned} \|u\|_{H^1(\gamma)} &\leq C |u|_{H^2(S_1)} \\ &\leq C |u|_{H^{2,2}_B(S)}. \end{aligned}$$

The scaling argument yields (3.2).  $\square$

Lemma 3.3. Let  $\gamma_i$  be the sides of  $T$ ,  $1 \leq i \leq 4$  (resp.  $1 \leq i \leq 3$ , for the right angle triangle). Assume that  $\gamma_1$  lies on  $x$ -axis and that  $v(x)$  is a polynomial degree  $p$  on  $\gamma_1$  and that  $v$  vanishes at the endpoints of  $\gamma_1$ . Then there exists a polynomial  $V(x,y)$  of degree  $p$  in  $x$  and degree 1 in  $y$  such that  $V = 0$  on  $\gamma_i$ , ( $i \neq 1$ ) and  $V = v$  on  $\gamma_1$ , and

$$(3.3) \quad \|V\|_{H^1(T)}^2 \leq Ch \|v\|_{H^1(\gamma)}^2$$

with  $C$  independent of  $h$ ,  $v$  and  $T$ .

Proof. Let  $T = [0,1] \times [0,1]$  and  $\gamma_1 = \{(x,0) \mid 0 \leq x \leq 1\}$ . Set

$$V(x,y) = v(x)(1-y).$$

Obviously  $V(x,1) = V(0,y) = V(1,y) = 0$  and  $V(x,0) = v(x)$ .  $V(x,y)$  is a polynomial of degree  $p$  in  $x$  and degree 1 in  $y$ , and

$$\|V\|_{H^1(S)}^2 \leq C(\|v\|_{L^2(\gamma_1)}^2 + |v|_{H^1(\gamma_1)}^2).$$

Since  $v(0) = v(1) = 0$ , we have

$$\|v\|_{L^2(\gamma_1)}^2 \leq C \|v\|_{H^1(\gamma_1)}^2.$$

Using the scaling argument we obtain (3.3) for the rectangle.  $\square$

The proof for triangle  $T$  can be found in [5], [7].

Lemma 3.4. The space  $H_{\beta}^{2,2}(T)$ ,  $\Phi_{\beta} = r^{\beta}$ ,  $0 < \beta < 1$  is compactly imbedded in the space  $H^1(T)$ .

Proof. Let  $u \in H_{\beta}^{2,2}(T)$  and assume first that  $u(0) = 0$ . Let  $(r, \theta)$  be the polar coordinates and  $v = r^{\beta} u$ . Then

$$v_{\theta}^{\alpha_2} = r^{\beta} u_{\theta}^{\alpha_2}, \quad 0 \leq \alpha_2 \leq 2,$$

$$v_{r\theta}^{\alpha_2} = r^{\beta} u_{r\theta}^{\alpha_2} + \beta r^{\beta-1} u_{\theta}^{\alpha_2}, \quad 0 \leq \alpha_2 \leq 1,$$

$$v_{rr}^2 = r^{\beta} u_{rr}^2 + 2\beta r^{\beta-1} u_r + \beta(\beta-1)r^{\beta-2} u.$$

Since  $u(0) = 0$  we have by Lemma 8 of [3] (or Lemma 4.3 of [5])

$$\|r^{\beta-1} u_r\|_{H^0(T)} \leq C \|u\|_{H_{\beta}^{2,2}(T)}$$

$$\|r^{\beta-2} u\|_{H^0(T)} \leq C \|r^{\beta-1} u_r\|_{H^0(T)} \leq C \|u\|_{H_{\beta}^{2,2}(T)}$$



where we denoted

$$D^1 u = \left( \sum_{|\alpha|=1} |D^\alpha u|^2 \right)^{1/2}.$$

This implies that  $v \in H^2(T)$ . Suppose now that  $\{u_j\}_{j=1}^\infty$  is a bounded sequence in  $H_\beta^{2,2}(T)$  and  $u_j(0) = 0$ . Then  $v_j = r^\beta u_j$   $j = 1, 2, \dots$  is uniformly bounded in  $H^2(T)$ . Because the space  $H^2(T)$  is compactly imbedded in the Sobolev space  $W^{1,q}(T)$  for any  $1 < q < \infty$ , there exists subsequence denoted again by  $\{v_j\}_{j=1}^\infty$  which converges to  $\tilde{v} \in W^{1,q}(T)$  in  $W^{1,q}(T)$  and  $\tilde{v}(0) = 0$ . Let  $\tilde{u} = r^{-\beta} \tilde{v}$ . Then

$$\tilde{u}_x = r^{-\beta} \tilde{v}_x - \beta r^{-\beta-1} \tilde{v} \frac{x}{r}$$

$$\tilde{u}_y = r^{-\beta} \tilde{v}_y - \beta r^{-\beta-1} \tilde{v} \frac{y}{r}$$

and since  $\tilde{v}(0)$  we have

$$\|r^{-\beta-1} \tilde{v}\|_{H^0(T)} \leq C \|r^{-\beta} D^1 \tilde{v}\|_{H^0(T)}.$$

Hence

$$\|D^1 \tilde{u}\|_{H^0(T)} \leq C \|r^{-\beta} D^1 \tilde{v}\|_{H^0(T)}$$

by Hölder inequality

$$\leq C \|r^{-\beta p}\|_{H^0(T)}^{\frac{1}{p}} \| (D^1 \tilde{v})^q \|_{H^0(T)}^{\frac{1}{q}}$$

$$\leq C \|r^{-\beta p}\|_{H^0(T)}^{\frac{1}{p}} \|\tilde{v}\|_{W^{1,2q}(T)}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p > 1$  and  $\beta p < 1$  and hence  $\tilde{u} \in H^1(T)$ .

Similarly we have

$$\|u_j - \tilde{u}\|_{H^1(T)} \leq C \|v_j - \tilde{v}\|_{W^{1,2q}(T)} \rightarrow 0 \text{ as } j \rightarrow \infty$$

and hence  $\{u_j\}$  is a convergent sequence in  $H^1(T)$ .

If  $u_j(0) \neq 0$  then we define  $\bar{u}_j = u_j - u_j(0)$ . Then using Lemma 2.1 we have

$$\|\bar{u}_j\|_{H_\beta^{2,2}(T)} \leq C \|u_j\|_{H_\beta^{2,2}(T)} \text{ and } |u_j(0)| \leq C \|u_j\|_{H_\beta^{2,2}(T)}.$$

Using the first part of the proof there is a subsequence, once more denoted by  $u_j$  such that  $u_j(0) \rightarrow A$  and  $\bar{u}_j \rightarrow \tilde{u}$  in  $H^1(T)$ . This completes the proof.  $\square$

**Lemma 3.5.** Let  $T$  be a triangle with vertices  $(x_i, y_i)$   $1 \leq i \leq 3$  and  $x_1 = y_1 = 0$ . If  $u \in H_\beta^{2,2}(T)$  with  $\phi_\beta = r^\beta$ ,  $0 \leq \beta < 1$ , ( $H_\beta^{2,2}(T) = H^2(T)$  for  $\beta = 0$ ), and  $v$  is the linear function interpolating  $u$  at vertices  $(x_i, y_i)$ ,  $1 \leq i \leq 3$ , then

$$(3.4) \quad \|u-v\|_{H^1(T)} \leq C h^{1-\beta} |u|_{H_\beta^{2,2}(T)}$$

and

$$(3.5) \quad \|u-v\|_{H_\beta^{2,2}(T)} \leq C |u|_{H_\beta^{2,2}(T)}$$

with  $C$  independent of  $u$  and  $T$ .

**Proof.** If  $\beta = 0$  then the lemma is standard [2], [11]. Let  $S$  be a standard

triangle  $\{(x,y) \mid 0 < y < 1-x, 0 < x < 1\}$  with vertices  $(x_i, y_i)$ ,  $1 \leq i \leq 3$ , and let  $U \in H_{\beta}^{2,2}(S)$  with  $\phi_{\beta} = r^{\beta}$  and  $V$  be the linear interpolation of  $U$  (which exists because of Lemma 2.1). We first prove that for a constant  $C$  independent of  $U$

$$(3.6) \quad \|U\|_{H_{\beta}^{2,2}(S)}^2 \leq C(\|U\|_{H_{\beta}^{2,2}(S)}^2 + \sum_{i=1}^3 |U(x_i, y_i)|^2).$$

Suppose that (3.6) is false, then there exists  $U_j \in H_{\beta}^{2,2}(S)$ ,  $j = 1, 2, \dots$  such that

$$(3.7) \quad \|U_j\|_{H_{\beta}^{2,2}(S)}^2 = 1$$

and

$$(3.8) \quad \|U_j\|_{H_{\beta}^{2,2}(S)}^2 + \sum_{i=1}^3 |U_j(x_i, y_i)|^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Since  $H^1(S)$  is compactly imbedded in  $H_{\beta}^{2,2}(S)$ , by (Lemma 3.4), there exists a subsequence denoted again by  $\{U_j\}$  which is convergent in  $H^1(S)$ . (3.8) shows that  $\{U_j\}$  is a Cauchy sequence in  $H_{\beta}^{2,2}(S)$ . By the argument similar to that in the previous lemma, we can show that there is a subsequence once more denoted by  $U_j$  that  $\{r^{\beta} U_j\}_{j=1}^{\infty}$  is a Cauchy sequence in  $H^2(S)$ ,  $r^{\beta} U_j \rightarrow V$  in  $H^2(S)$ , and  $U_j \rightarrow \bar{U} = r^{-\beta} V \in H_{\beta}^{2,2}(S)$  as  $j \rightarrow \infty$ . Since  $\|U_j\|_{H_{\beta}^{2,2}(S)} \rightarrow 0$ ,  $D^{\alpha} \bar{U} = 0$  for  $|\alpha| = 2$ . Therefore  $\bar{U}$  is a linear function of  $S$ . Because  $H_{\beta}^{2,2}(S) \subset C^0(\bar{S})$  by Lemma 2.1 and  $U_j(x_i, y_i) \rightarrow 0$  as  $j \rightarrow \infty$ ,  $1 \leq i \leq 3$ ,  $\bar{U}$  vanishes at vertices  $(x_i, y_i)$ . Hence the linear function  $\bar{U} \equiv 0$  on  $S$ , which is a contradiction to (3.7).

Applying (3.6) to  $(U - V)$  we have

$$\|U - V\|_{H_{\beta}^{2,2}(S)} \leq C|U - V|_{H_{\beta}^{2,2}(S)} = C|U|_{H_{\beta}^{2,2}(S)}$$

and

$$\|U - V\|_{H^1(S)} \leq \|U - V\|_{H_{\beta}^{2,2}(S)} \leq C|U|_{H_{\beta}^{2,2}(S)}.$$

By the standard scaling and mapping argument, we get (3.6) and (3.7).  $\square$

Lemma 3.6. Let  $T$  be a rectangle with vertices  $(x_i, y_i)$ ,  $1 \leq i \leq 4$  and  $x_1 = y_1 = 0$ . If  $u \in H_{\beta}^{2,2}(T)$  with  $\phi_{\beta} = r$ ,  $0 \leq \beta < 1$  ( $H_{\beta}^{2,2}(T) = H^2(T)$  for  $\beta = 0$ ) and  $v$  is the bilinear function interpolating  $u$  at vertices  $(x_i, y_i)$ ,  $1 \leq i \leq 4$ , then

$$(3.9) \quad \|u - v\|_{H^1(T)} \leq Ch^{1-\beta} |u|_{H_{\beta}^{2,2}(T)}$$

and

$$(3.10) \quad \|u - v\|_{H_{\beta}^{2,2}(T)} \leq C|u|_{H_{\beta}^{2,2}(T)}$$

with  $C$  independent of  $u$  and  $T$ .

Proof. Let  $S = (0,1) \times (0,1)$  with vertices  $(x_i, y_i)$ ,  $1 \leq i \leq 4$ ,  $U \in H_{\beta}^{2,2}(S)$  with  $\phi_{\beta} = r$ . Let  $V$  be the bilinear function interpolating  $U$  at  $(x_i, y_i)$ ,  $1 \leq i \leq 4$ . It can be proven analogously as for (3.6) that for some constant  $C$  independent of  $U$

$$(3.11) \quad \|U\|_{H_{\beta}^{2,2}(S)}^2 \leq C(|U|_{H_{\beta}^{2,2}(S)}^2 + \sum_{i=1}^4 |U(x_i, y_i)|^2).$$

Let  $W$  be a linear function interpolating  $U$  at 3 vertices of  $S$  other than the origin and  $\phi$  be the bilinear function which is equal to 1 at the origin and vanishes at other vertices of  $S$ . Let  $Z = U - W$ , then  $V = W + Z(0,0)\phi$  and

$$\|U-V\|_{H_{\beta}^{2,2}(S)} \leq \|U-W\|_{H_{\beta}^{2,2}(S)} + |Z(0,0)| \|\phi\|_{H_{\beta}^{2,2}(S)}$$

by the imbedding theorem

$$\leq C\|U-W\|_{H_{\beta}^{2,2}(S)}.$$

In the same way as in Lemma 3.5 we have

$$\|U-W\|_{H_{\beta}^{2,2}(S)} \leq C|U|_{H_{\beta}^{2,2}(S)}$$

which implies (3.9) and (3.10) by the standard scaling argument.  $\square$

## 4. PROPERTIES OF THE POLYNOMIAL APPROXIMATION

In this section we shall analyze the approximation properties of the polynomials on a square, rectangle and parallelogram.

Lemma 4.1. Let  $S = (-1,1) \times (-1,1)$  be square domain and

$u(x) = \sum_{i,j=0}^{\infty} c_{i,j} p_i(x_1) p_j(x_2)$ , where  $p_i(x_1)$  (resp.  $p_j(x_2)$ ) are Legendre polynomials on  $I_{x_1} = (-1,1)$  (resp.  $I_{x_2} = (-1,1)$ ). Then

$$(4.1) \quad \int_S |D^\alpha u|^2 (1-x_1^2)^{\alpha_1} (1-x_2^2)^{\alpha_2} dx = \sum_{\substack{i \geq \alpha_1 \\ j \geq \alpha_2}} c_{i,j}^2 \frac{2}{2i+1} \cdot \frac{2}{2j+1} \frac{(i+\alpha_1)!(j+\alpha_2)!}{(i-\alpha_1)!(j-\alpha_2)!}$$

provided that the left or right hand side is finite.

Proof. Using the basic properties of Legendre polynomials we get for  $m = 1, 2$

$$\int_{-1}^1 p_j^{(\alpha)}(x_m) p_i^{(\alpha)}(x_m) \cdot (1-x_m^2)^{\alpha_m} dx_m = \begin{cases} \frac{2}{2i+1} \cdot \frac{(i+\alpha_m)!}{(i-\alpha_m)!} & \text{for } \alpha_m \leq i, j \quad i = j \\ 0 & \text{for } \alpha_m > i, j \quad \text{or } i \neq j \end{cases}$$

and the lemma easily follows.  $\square$

Lemma 4.2. If  $u \in H^{k+3}(S)$ ,  $k = \max(k_1, k_2)$ ,  $k_1, k_2 \geq 2$ , then there exists a

polynomial  $\phi(x_1, x_2) = \sum_{\substack{0 \leq i \leq k_1 \\ 0 \leq j \leq k_2}} d_{i,j} x_1^i x_2^j$  such that for  $0 \leq m \leq 2$

$$\begin{aligned}
 (4.2) \quad \|D^m(u-\phi)\|_{H^0(S)}^2 \leq & C \left\{ \frac{(k_1-s_1)!}{(k_1+s_1+2-2m)!} \sum_{\ell=0}^2 \left\| \frac{\partial^{s_1+1+\ell} u}{\partial x_1^{s_1+1} \partial x_2^\ell} \right\|_{H^0(S)}^2 \right. \\
 & \left. + \frac{(k_2-s_2)!}{(k_2+s_2+2-2m)!} \sum_{\ell=0}^2 \left\| \frac{\partial^{s_2+1+\ell} u}{\partial x_1^\ell \partial x_2^{s_2+1}} \right\|_{H^0(S)}^2 \right\}
 \end{aligned}$$

where  $s_i$  is any integer,  $1 \leq s_i \leq k_i$ , ( $i = 1, 2$ ), and  $C$  is independent of  $k_i$ . Moreover

$$(4.3) \quad u = \phi \text{ at the vertices of } S.$$

Proof. Let  $p_i(x_1)$  and  $p_j(x_2)$  be the same as in the previous lemma.

Since  $u \in H^5(S)$  we have the following Fourier expansion:

$$\begin{aligned}
 u_{x_1 x_2} &= \sum_{i,j=0}^{\infty} c_{i,j} p_i(x_1) p_j(x_2), \\
 u_{x_1 x_2}(x_1, -1) &= \sum_{i=0}^{\infty} a_i p_i(x_1), \\
 u_{x_1 x_2}(-1, x_2) &= \sum_{j=0}^{\infty} b_j p_j(x_2), \\
 u_{x_1}(x_1, -1) &= \sum_{i=0}^{\infty} d_i p_i(x_1), \\
 u_{x_2}(-1, x_2) &= \sum_{j=0}^{\infty} e_j p_j(x_2).
 \end{aligned}$$

Set

$$\chi_1(x_1, x_2) = \sum_{\substack{0 \leq i \leq k_1 - 2 \\ 0 \leq j \leq k_2 - 2}} c_{i,j} p_i(x_1) p_j(x_2),$$

$$\chi_2(x_1) = \sum_{0 \leq i \leq k_1 - 2} a_i p_i(x_1), \quad \tilde{\chi}_2(x_1) = \int_{-1}^{x_1} \chi_2(\xi_1) d\xi_1 + u_{x_1 x_2}(-1, -1),$$

$$\chi_3(x_2) = \sum_{0 \leq j \leq k_2 - 2} b_j p_j(x_2), \quad \tilde{\chi}_3(x_2) = \int_{-1}^{x_2} \chi_3(\xi_2) d\xi_2 + u_{x_1 x_2}(-1, -1),$$

$$\chi_4(x_1) = \sum_{0 \leq i \leq k_1 - 2} d_i p_i(x_1), \quad \tilde{\chi}_4(x_1) = \int_{-1}^{x_1} \chi_4(\xi_1) d\xi_1 + u_{x_1}(-1, -1),$$

$$\chi_5(x_2) = \sum_{0 \leq j \leq k_2 - 2} e_j p_j(x_2), \quad \tilde{\chi}_5(x_2) = \int_{-1}^{x_2} \chi_5(\xi_2) d\xi_2 + u_{x_2}(-1, -1).$$

Let

$$\phi(x) = \phi_1 + \phi_2,$$

where

$$(4.4) \quad \phi_1 = \int_{-1}^{x_1} \int_{-1}^{x_2} \int_{-1}^{\xi_1} \int_{-1}^{\xi_2} \chi_1(\xi_1, \xi_2) d\xi_1 d\xi_2 d\xi_1 d\xi_2 + (1+x_2) \int_{-1}^{x_1} \tilde{\chi}_2(\xi_1) d\xi_1 \\ + \int_{-1}^{x_1} \tilde{\chi}_4(\xi_1) d\xi_1$$

$$(4.5) \quad \phi_2 = (1+x_1) \int_{-1}^{x_2} \tilde{\chi}_3(\xi_2) d\xi_2 + \int_{-1}^{x_2} \tilde{\chi}_5(\xi_2) d\xi_2 + u(-1, -1) + (1+x_1)(1+x_2) u_{x_1 x_2}(-1, -1).$$

The degrees of  $\phi_i$  ( $i = 1, 2$ ) in  $x_1$  and  $x_2$  are at most  $k_1$  and  $k_2$  respectively. We can readily verify that



$$\begin{aligned}
u(x_1, x_2) = & \int_{-1}^{x_1} \int_{-1}^{x_2} \int_{-1}^{\xi_1} \int_{-1}^{\xi_2} u_{x_1 x_2}(\xi_1, \xi_2) d\tilde{\xi}_1 d\tilde{\xi}_2 d\xi_1 d\xi_2 + (1+x_1) \int_{-1}^{x_2} u_{x_1 x_2}(-1, \xi_2) d\xi_2 \\
& + (1+x_2) \int_{-1}^{x_1} u_{x_1 x_2}(\xi_1, -1) d\xi_1 + \int_{-1}^{x_1} u_{x_1}(\xi_1, -1) d\xi_1 + \int_{-1}^{x_2} u_{x_2}(-1, \xi_2) d\xi_2 + u(-1, -1) \\
& - (1+x_1)(1+x_2) u_{x_1 x_2}(-1, -1).
\end{aligned}$$

Let

$$u = u_1 + u_2$$

with

$$\begin{aligned}
(4.6) \quad u_1 = & \int_{-1}^{x_1} \int_{-1}^{x_2} \int_{-1}^{\xi_1} \int_{-1}^{\xi_2} \left( \sum_{\substack{0 \leq i \leq k_1 - 2 \\ 0 \leq j \leq k_2 - 2}} + \sum_{j \geq k_1 - 1} \right) (c_{i,j} p_i(\tilde{\xi}_1) p_j(\tilde{\xi}_2)) d\tilde{\xi}_1 d\tilde{\xi}_2 d\xi_1 d\xi_2 \\
& + (1+x_2) \int_{-1}^{x_1} u_{x_1 x_2}(\xi_1, -1) d\xi_1 + \int_{-1}^{x_1} u_{x_1}(\xi_1, -1) d\xi_1,
\end{aligned}$$

$$\begin{aligned}
(4.7) \quad u_2 = & \int_{-1}^{x_1} \int_{-1}^{x_2} \int_{-1}^{\xi_1} \int_{-1}^{\xi_2} \sum_{\substack{0 \leq i \leq k_1 - 2 \\ j \geq k_2 - 1}} c_{i,j} p_i(\tilde{\xi}_1) p_j(\tilde{\xi}_2) d\tilde{\xi}_1 d\tilde{\xi}_2 d\xi_1 d\xi_2 \\
& + (1+x_1) \int_{-1}^{x_2} u_{x_1 x_2}(-1, \xi_2) d\xi_2 + \int_{-1}^{x_2} u_{x_2}(-1, \xi_2) d\xi_2 + u(-1, -1) \\
& - (1+x_1)(1+x_2) u_{x_1 x_2}(-1, -1);
\end{aligned}$$

we get

$$u - \phi = u_1 - \phi_1 + u_2 - \phi_2$$

$$u_1 - \phi_1 = F_1 + F_2 + F_3, \quad u_2 - \phi_2 = F_4 + F_5 + F_6,$$

$$F_1 = \int_{-1}^{x_1} \int_{-1}^{x_2} \int_{-1}^{\xi_1} \int_{-1}^{\xi_2} \sum_{\substack{i \geq k_1 - 1 \\ j \geq 0}} c_{i,j} p_i(\tilde{\xi}_1) p_j(\tilde{\xi}_2) d\tilde{\xi}_1 d\tilde{\xi}_2 d\xi_1 d\xi_2$$

$$F_4 = \int_{-1}^{x_1} \int_{-1}^{x_2} \int_{-1}^{\xi_1} \int_{-1}^{\xi_2} \sum_{\substack{0 \leq i \leq k_1 - 2 \\ j \geq k_2 - 1}} c_{i,j} p_i(\tilde{\xi}_1) p_j(\tilde{\xi}_2) d\tilde{\xi}_1 d\tilde{\xi}_2 d\xi_1 d\xi_2,$$

$$F_2 = (1+x_2) \int_{-1}^{x_2} \int_{-1}^{\xi_1} \sum_{i \geq k_1 - 1} a_i p_i(\tilde{\xi}_1) d\tilde{\xi}_1,$$

$$F_5 = (1+x_1) \int_{-1}^{x_2} \int_{-1}^{\xi_2} \sum_{j \geq k_2 - 1} b_j p_j(\tilde{\xi}_2) d\tilde{\xi}_2 d\xi_2,$$

$$F_3 = \int_{-1}^{x_1} \int_{-1}^{\xi_1} \sum_{i \geq k_1 - 1} d_i p_i(\tilde{\xi}_1) d\tilde{\xi}_1 d\xi_1,$$

$$F_6 = \int_{-1}^{x_2} \int_{-1}^{\xi_2} \sum_{j \geq k_2 - 1} e_j p_j(\tilde{\xi}_2) d\tilde{\xi}_2 d\xi_2.$$

It can be seen that

$$\begin{aligned} \frac{\partial^2 F_1}{\partial x_1^2} &= \int_{-1}^{x_2} \int_{-1}^{\xi_2} \sum_{\substack{i \geq k_1 - 1 \\ 0 \leq j < \infty}} c_{i,j} p_i(x_1) p_j(\tilde{\xi}_2) d\tilde{\xi}_2 d\xi_2 \\ &= \sum_{\substack{i \geq k_1 - 1 \\ 0 \leq j < \infty}} c_{i,j} p_i(x_1) \frac{1}{(2j+1)} \left( \frac{p_{j+2}(x_2) - p_j(x_2)}{2j+3} - \frac{p_j(x_2) - p_{j-2}(x_2)}{2j-1} \right) \end{aligned}$$

and

$$\begin{aligned}
\left\| \frac{\partial^2 F_1}{\partial x_1^2} \right\|_{H^0(S)}^2 &\leq C \sum_{\substack{i \geq k_1-1 \\ j \geq 0}} \frac{2}{2i+1} \frac{2}{(2j+1)} \frac{2}{(2j+1)^4} \\
&\leq C \frac{(k_1-s_1)!}{(k_1+s_1-2)!} \sum_{\substack{i \geq s_1-1 \\ j \geq 0}} |c_{i,j}|^2 \frac{2}{2i+1} \frac{2}{2j+1} \frac{(i+s_1-1)!}{(i-(s_1-1))!}
\end{aligned}$$

by Lemma 4.1

$$\leq C \frac{(k_1-s_1)!}{(k_1+s_1-2)!} \left\| \frac{\partial^{s_1+3} u}{\partial x_1^{s_1+1} \partial x_2^2} \right\|_{H^0(S)}^2.$$

In the same way we have with  $C$  independent of  $k$

$$(4.8) \quad \|D^m F_1\|_{H^0(S)}^2 \leq C \frac{(k_1-s_1)!}{(k_1+s_1+2-2m)!} \left\| \frac{\partial^{s_1+3} u}{\partial x_1^{s_1+1} \partial x_2^2} \right\|_{H^0(S)}^2, \quad 0 \leq m \leq 2$$

and

$$(4.9) \quad \|D^m F_4\|_{H^0(S)}^2 \leq C \frac{(k_2-s_2)!}{(k_2+s_2+2-2m)!} \left\| \frac{\partial^{s_2+3} u}{\partial x_1^2 \partial x_2^{s_2+1}} \right\|_{H^0(S)}^2, \quad 0 \leq m \leq 2.$$

Next we estimate  $F_2$ . Since

$$\frac{\partial^2 F_2}{\partial x_1^2} = \left( \sum_{i \geq k_1-1} a_i p_i(x_1) \right) (1+x_2)$$

$$\begin{aligned}
\left\| \frac{\partial^2 F_2}{\partial x_1^2} \right\|_{H^0(I_{x_1})}^2 &\leq C \sum_{i \geq k_1-1} a_i^2 \frac{2}{2i+1} \leq C \frac{(k_1-s_1)!}{(k_1+s_1-2)!} \sum_{i \geq s_1-1} a_i^2 \frac{2}{2i+1} \frac{(i+s_1-1)!}{(i-(s_1-2))!} \\
&\leq C \frac{(k_1-s_1)!}{(k_1+s_1-2)!} \left\| \frac{\partial^{s_1+2} u}{\partial x_1^{s_1+1} \partial x_2} \right\|_{H^0(I_{x_1})}^2
\end{aligned}$$

we obtain

$$\left\| \frac{\partial^2 F_2}{\partial x_1^2} \right\|_{H^0(S)}^2 \leq C \frac{(k_1 - s_1)!}{(k_1 + s_1 - 2)!} \left\| \frac{\partial^{s_1+2} u}{\partial x_1^{s_1+1} \partial x_2} \right\|_{H^0(S)}^2.$$

In a similar way we have

$$\left\| \frac{\partial F_2}{\partial x_1} \right\|_{H^0(S)}^2 \leq C \frac{(k_1 - s_1)!}{(k_1 + s_1)!} \left\| \frac{\partial^{s_1+2} u}{\partial x_1^{s_1+1} \partial x_2} \right\|_{H^0(S)}^2,$$

$$\|F_2\|_{H^0(S)}^2 \leq C \frac{(k_1 - s_1)!}{(k_1 + s_1 + 2)!} \left\| \frac{\partial^{s_1+2} u}{\partial x_1^{s_1+1} \partial x_2} \right\|_{H^0(S)}^2.$$

It is obvious that  $\frac{\partial^2 F_2}{\partial x_2^2} = 0$  and  $\left\| \frac{\partial F_2}{\partial x_2} \right\|_{H^0(S)} \leq C \|F_2\|_{H^0(S)}$ . Therefore for  
for  $0 \leq m \leq 2$

$$(4.10) \quad \|D^m F_2\|_{H^0(S)}^2 \leq C \frac{(k_1 - s_1)!}{(k_1 + s_1 + 2 - 2m)!} \left\| \frac{\partial^{s_1+2} u}{\partial x_1^{s_1+1} \partial x_2} \right\|_{H^0(S)}^2.$$

Analogously we have for  $0 \leq m \leq 2$

$$(4.11) \quad \|D^m F_5\|_{H^0(S)}^2 \leq C \frac{(k_2 - s_2)!}{(k_1 + s_2 + 2 - 2m)!} \left\| \frac{\partial^{s_2+2} u}{\partial x_1 \partial x_2^{s_2+1}} \right\|_{H^0(S)}^2,$$

$$(4.12) \quad \|D^m F_3\|_{H^0(S)}^2 \leq C \frac{(k_1 - s_1)!}{(k_1 + s_1 + 2 - 2m)!} \left\| \frac{\partial^{s_1+1} u}{\partial x_1^{s_1+1}} \right\|_{H^0(S)}^2,$$

$$(4.13) \quad \|D^m F_6\|_{H^0(S)}^2 \leq C \frac{(k_2 - s_2)!}{(k_1 + s_2 - 2 - 2m)!} \left\| \frac{\partial^{s_2+1} u}{\partial x_2^{s_2+1}} \right\|_{H^0(S)}^2.$$

Hence using (4.8), (4.10) and (4.12) we get

(4.14)

$$\|D^m(u_1 - \phi_1)\|_{H^0(S)}^2 \leq C \frac{(k_1 - s_1)!}{(k_1 + s_1 + 2 - 2m)!} \sum_{\ell=0}^2 \left\| \frac{\partial^{s_1+1+\ell} u}{\partial x_1^{s_1+1} \partial x_2^\ell} \right\|_{H^0(S)}^2, \quad 0 \leq m \leq 2$$

and by (4.9), (4.11) and (4.13)

(4.15)

$$\|D^m(u_2 - \phi_2)\|_{H^0(S)}^2 \leq C \frac{(k_2 - s_2)!}{(k_2 + s_2 + 2 - 2m)!} \sum_{\ell=0}^2 \left\| \frac{\partial^{s_2+1+\ell} u}{\partial x_1^{\ell} \partial x_2^{s_2+1+\ell}} \right\|_{H^0(S)}^2, \quad 0 \leq m \leq 2.$$

(4.14) and (4.15) yield (4.2). From the orthogonality of Legendre polynomials

$$F_1(\pm 1, x_2) = F_4(x_1, \pm 1) = F_2(\pm 1, x_2) = F_5(x_1, \pm 1) = F_3(\pm 1) = F_6(\pm 1) = 0$$

which implies (4.3).  $\square$

By the scaling argument we obtain immediately the following lemma

Lemma 4.3. Let  $\Omega = (a, b) \times (c, d)$  with  $h_1 = (b-a)$  and  $h_2 = (d-c)$ .

If  $u \in H^{k+3}(\Omega)$ ,  $k = \max(k_1, k_2)$ ,  $k_1, k_2 \geq 2$ , then there is a polynomial

$$\phi(x) = \sum_{\substack{0 \leq i \leq k_1 \\ 0 \leq j \leq k_2}} c_{i,j} x_1^i x_2^j \text{ such that for any integers } 1 \leq s_i \leq k_i, \quad i = 1, 2, \text{ and}$$

$$0 \leq m \leq 2$$

(4.16)

$$\begin{aligned} \|D^m(u - \phi)\|_{H^0(\Omega)}^2 &\leq C h^{-2m} \left\{ \frac{(k_1 - s_1)!}{(k_1 + s_1 + 2 - 2m)!} \left(\frac{h_1}{2}\right)^{2(s_1+1)} \sum_{\ell=0}^2 \left\| \frac{\partial^{s_1+1+\ell} u}{\partial x_1^{s_1+1} \partial x_2^\ell} \right\|_{H^0(\Omega)}^2 \left(\frac{h_2}{2}\right)^{2\ell} \right. \\ &\quad \left. + \frac{(k_2 - s_2)!}{(k_2 + s_2 + 2 - 2m)!} \left(\frac{h_2}{2}\right)^{2(s_2+1)} \sum_{\ell=0}^2 \left\| \frac{\partial^{s_2+1+\ell} u}{\partial x_1^\ell \partial x_2^{s_2+1+\ell}} \right\|_{H^0(\Omega)}^2 \left(\frac{h_1}{2}\right)^{2\ell} \right\} \end{aligned}$$

where  $\underline{h} = \min(h_1, h_2)$  and  $C$  is independent of  $k$ . Moreover

$$(4.17) \quad u = \phi \quad \text{at the vertices of } \Omega. \quad \square$$

**Theorem 4.1.** Let  $\Omega_1 = (a, b) \times (c, d) \subset \subset \Omega$  with  $h_1 = (b-a) \leq \lambda_1 r_0$  and  $h_2 = (d-c) \leq \lambda_2 r_0$ ,  $\lambda_i \geq 0$ , ( $i = 1, 2$ ) and  $r_0$  be the distance between origin and  $\Omega_1$ . Assume that  $h/\underline{h} = \max(h_1, h_2)/\min(h_1, h_2) \leq \kappa$ . If  $u \in H_{\beta}^{k+3,2}(\Omega)$  with  $\phi_{\beta} = r^{\beta}$ ,  $0 < \beta < 1$ , then there exists a polynomial  $\phi = \sum_{\substack{0 \leq i \leq k_1 \\ 0 \leq j \leq k_2}} c_{i,j} x_1^i x_2^j$  on  $\Omega_1$  with  $2 \leq k_1, k_2 \leq k$  such that for  $0 \leq m \leq 2$

$$(4.18) \quad \|D^m(u-\phi)\|_{H^0(\Omega_1)}^2 \leq C r_0^{2(2-m-\beta)} \sum_{j=1}^2 \frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3 - 2m)} \left(\frac{\lambda_j}{2}\right)^{2s_j} \|u\|_{H_{\beta}^{s_j+3,2}(\Omega)}$$

where  $s_j$  is any real number  $1 \leq s_j \leq k_j$ ,  $j = 1, 2$ , and  $H_{\beta}^{s_j+3,2}(\Omega)$  is the interpolation space  $(H_{\beta}^{\tilde{k}_j+2,2}(\Omega), H_{\beta}^{\tilde{k}_j+3,2}(\Omega))_{\theta_j, \infty}$  for some integer  $\tilde{k}_j = s_j + 1 - \theta_j \leq k_j$ ,  $0 \leq \theta_j \leq 1$  and  $C$  is independent of  $k$ , but dependent on  $\lambda_j$  and  $s_j$ ,  $j = 1, 2$ . Moreover

$$(4.19) \quad u = \phi \quad \text{at the vertices of } \Omega_1.$$

**Proof.** Applying Lemma 4.3, by (4.14) and (4.15) we have for the integer  $\tilde{k}_j$ ,  $1 \leq \tilde{k}_j \leq k_j$ ,  $j = 1, 2$

$$\begin{aligned} \|D^m(u_1 - \phi_1)\|_{H^0(\Omega_1)}^2 &\leq Ch^{-2m} \frac{(k_1 - \tilde{k}_1)!}{(k_1 + \tilde{k}_1 + 2 - 2m)!} \left(\frac{h_1}{2}\right)^{2(\tilde{k}_1+1)} \sum_{\ell=0}^2 \|u\|_{x_1}^2 \tilde{k}_1^{\ell+1} \|x_2^\ell\|_{H^0(\Omega_1)}^2 \left(\frac{h_2}{2}\right)^{2\ell} \\ &\leq Cr_0^{2(2-m-\beta)} \frac{(k_1 - \tilde{k}_1)!}{(k_1 + \tilde{k}_1 + 2 - 2m)!} \left(\frac{\lambda_1}{2}\right)^{2\tilde{k}_1} \|u\|_{H_\beta}^2 \tilde{k}_1^{+3,2}(\Omega) \end{aligned}$$

and similarly

$$\|D^m(u_2 - \phi_2)\|_{H^0(\Omega)}^2 \leq Cr_0^{2(2-m-\beta)} \frac{(k_2 - \tilde{k}_2)!}{(k_2 + \tilde{k}_2 + 2 - 2m)!} \left(\frac{\lambda_2}{2}\right)^{2\tilde{k}_2} \|u\|_{H_\beta}^2 \tilde{k}_2^{+3,2}(\Omega).$$

Let

$$T_j u = D^m(u_j - \phi_j), \quad j = 1, 2.$$

Thus  $T_j$  is an operator:  $H_\beta^{\tilde{k}_j+3,2}(\Omega) \rightarrow H^m(\Omega_1)$  and the norm of the operator is bounded

$$\|T_j\|_{H_\beta^{\tilde{k}_j+3,2}(\Omega), H^m(\Omega_1)}^2 \leq Cr_0^{2(2-m-\beta)} \frac{(k_j - \tilde{k}_j)!}{(k_j + \tilde{k}_j + 2 - 2m)!} \left(\frac{\lambda_j}{2}\right)^{2\tilde{k}_j}.$$

If  $s_j = \tilde{k}_j - 1 + \theta_j$ ,  $0 < \theta_j < 1$ ,  $H_\beta^{s_j+3,2}(\Omega)$  is defined as interpolation space [10]

$$H_\beta^{s_j+3,2}(\Omega) = (H_\beta^{k_j+2,2}(\Omega), H_\beta^{k_j+3,3}(\Omega))_{\theta_j, \infty}.$$

$T_j$  is also linear and continuous operator:  $H_\beta^{s_j+3,2}(\Omega) \rightarrow H^m(\Omega_1)$  and

$$\begin{aligned} \|T_j\|_{H_\beta^{s_j+3,2}(\Omega), H^m(\Omega_1)}^2 &\leq \|T_j\|_{H_\beta^{k_j+2,2}(\Omega), H^m(\Omega_1)}^{2(1-\theta)} \cdot \|T_j\|_{H_\beta^{k_j+3,3}(\Omega), H^m(\Omega_1)}^{2\theta} \\ &\leq Cr_0^{2(2-m-\beta)} \left(\frac{\lambda_j}{2}\right)^{2s_j} \frac{(k_j - \tilde{k}_j)!}{(k_j + \tilde{k}_j + 2 - 2m)!} \frac{(k_j - \tilde{k}_j + 1)^{1-\theta}}{(k_j + \tilde{k}_j + 2 - 2m)^\theta}. \end{aligned}$$

For any integer  $n \geq 0$  and  $0 < \theta < 1$ , there are constants  $C_1, C_2 > 0$  independent of  $n$  such that [18, p.937]

$$(4.20) \quad C_1 \Gamma(n+1+\theta) \leq n!(n+1)^\theta \leq C_2 \Gamma(n+1+\theta).$$

Hence we have for  $j = 1, 2$

$$\|T_j\|_{H_\beta^{s_j+3,2}(\Omega), H^m(\Omega_1)}^2 \leq Cr_0^{2(2-m-\beta)} \left(\frac{\lambda_j}{2}\right)^{2s_j} \frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3 - 2m)}$$

and

$$\|D^m(u-\phi)\|_{H^0(\Omega_1)}^2 \leq Cr_0^{2(2-m-\beta)} \sum_{j=1}^2 \frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3 - 2m)} \left(\frac{\lambda_j}{2}\right)^{2s_j} \|u\|_{H_\beta^{s_j+3,2}(\Omega)}^2.$$

(4.19) follows now from (4.17).  $\square$

Let  $\Omega$  be a parallelogram with interior angle  $\omega$ ,  $0 < \omega < \frac{\pi}{2}$  and two sides  $a$  and  $b$  (see Fig. 4.1). We introduce the mapping

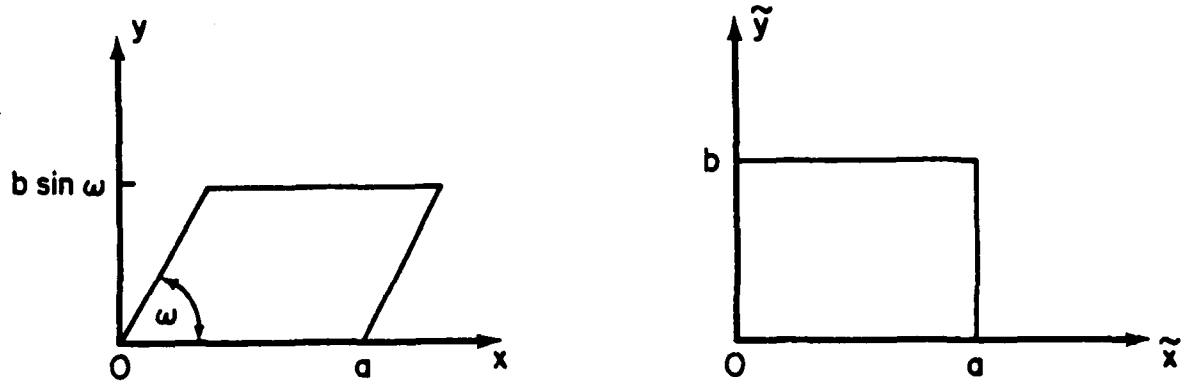


Fig. 4.1

Parallelogram



$$\begin{cases} \tilde{x}_1 = x_1 - x_2 \cot \omega \\ \tilde{x}_2 = x_2 / \sin \omega \end{cases}$$

and have analogous lemma and theorem for a parallelogram domain.

**Lemma 4.4.** Let  $\Omega$  be a parallelogram shown in Fig. 4.1. If  $u \in H^{k+3}(\Omega)$ ,  $k = \max(k_1, k_2)$ ,  $k_1, k_2 \geq 2$ , then there exists a polynomial

$$\phi(x_1, x_2) = \sum_{\substack{0 \leq i \leq k_1 \\ 0 \leq j \leq k_2}} c_{i,j} \chi_1^i \chi_2^j \text{ such that for } 0 \leq m \leq 2$$

$$(4.21) \quad \|D^m(u - \phi)\|_{H^0(\Omega)}^2$$

$$\leq C(\min(a, b))^{-2m} \left\{ \frac{(k_1 - s_1)!}{(k_1 + s_1 + 2 - 2m)!} \left(\frac{a}{2}\right)^{2(s_1+1)} \sum_{\ell=0}^2 \sum_{\ell'=0}^{\ell} \|u\|_{x_1^{s_1+1+\ell}, x_2^{\ell-\ell'}}^2_{H^0(\Omega)} \left(\frac{b}{2}\right)^{2\ell} \right. \\ \left. + \frac{(k_2 - s_2)!}{(k_2 + s_2 + 2 - 2m)!} b^{2(s_2+1)} \sum_{\ell=0}^2 \sum_{\ell'=0}^{s_2+1} \|u\|_{x_1^{\ell+\ell'}, x_2^{s_2+1-\ell'}}^2_{H^0(\Omega)} \left(\frac{a}{2}\right)^{2\ell} \right\}$$

where  $\chi_1 = (x_1 - x_2 \cos \omega)$ ,  $\chi_2 = x_2$ ,  $s_i$  is integer  $\leq k_i$ ,  $i = 1, 2$  and  $C$  is independent of  $k$ . Moreover

$$(4.22) \quad u = \phi \text{ at the vertices of } \Omega.$$

□

**Theorem 4.2.** Let  $\Omega_1$  be parallelogram  $\subset \subset \Omega$  with interior angle  $\omega$ ,  $0 < \omega < \frac{\pi}{2}$ , and let  $a$  and  $b$  denote the lengths of its two sides,  $a \leq \lambda_1 r_0$  and  $b \leq \lambda_2 r_0$  where  $r_0$  is the distance between  $\Omega_1$  and the origin where the vertex of  $\Omega$  is located. If  $u \in H^{k+3,2}_\beta(\Omega)$  with  $\phi_\beta = r^\beta$ ,  $0 \leq \beta < 1$  then there exists a polynomial  $\phi = \sum_{\substack{0 \leq i \leq k_1 \\ 0 \leq j \leq k_2}} c_{i,j} \chi_1^i \chi_2^j$ ,  $2 \leq k_1, k_2 \leq k$  such that

$$(4.23) \quad \|D^m(u-\phi)\|_{H^0(\Omega)}^2 \leq C r_0^{2(2-m-\beta)} \sum_{j=1}^2 \frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3 - 2m)} \left(\frac{\lambda_j}{2^{2-j}}\right)^{2s_j} \|u\|_{H_\beta^{s_j+3,2}(\Omega)}^2$$

where  $s_j$  is any real number  $1 \leq s_j \leq k_j$  ( $j = 1, 2$ ),  $C$  is some constant independent of  $k$  but dependent of angle,  $H_\beta^{s_1+3,2}(\Omega)$  is the interpolation space  $(H_\beta^{\tilde{k}_j+2,2}(\Omega), H_\beta^{\tilde{k}_j+3,2}(\Omega))_{\theta, \infty}$ ,  $0 \leq \theta \leq 1$ , for integer  $\tilde{k}_j \leq k_j$  such that  $s_j = \tilde{k}_j - 1 + \theta$ . Moreover

$$(4.24) \quad u = \phi \text{ at the vertices of } \Omega_1.$$

□

# 5. APPROXIMATION PROPERTIES OF PIECEWISE POLYNOMICAL ON A SQUARE AND A PARALLELOGRAM

Let  $\Omega$  be a square or a parallelogram and  $u \in B_{\beta,d}^2(\Omega)$  with the singularities in the vertices of the domain. We shall show that h-p version leads to an exponential rate of convergence if the mesh and the distribution of the degree are properly chosen.

Let us investigate the approximation of the function  $u \in B_{\beta,d}^2(\Omega)$  where  $\Omega = (0,1) \times (0,1)$  and  $\phi_\beta = r^\beta$ . Let a mesh on  $\Omega$  be as follows: let  $x_0 = y_0 = 0$ , and  $x_j = y_j = \sigma^{n+1-j}$ ,  $1 \leq j \leq n+1$  for  $0 < \sigma < 1$  and

$$\Omega_{1,j} = (x_{j-1}, x_j) \times (y_{j-1}, y_j) \quad \text{for } 1 \leq j \leq n+1,$$

$$\Omega_{2,j} = (x_{j-1}, x_j) \times (0, y_{j-1}) \quad \text{for } 2 \leq j \leq n+1,$$

$$\Omega_{3,j} = (0, x_{j-1}) \times (y_{j-1}, y_j) \quad \text{for } 2 \leq j \leq n+1.$$

The nodal points which are marked \* in the Fig. 5.1 are classified as irregular nodal points and the others as regular nodal points. The rectangulation

$\{\Omega_{i,j}, i = 1 \text{ for } j = 1, 1 \leq i \leq 3 \text{ for } 1 < j \leq n+1\}$  is denoted by  $\Omega_\sigma^{*n}$ .

If we divide  $\Omega_{i,j}$  for  $i \geq 2, j > 2$  into three triangles  $\Omega_{i,j}^k$ ,

$1 \leq k \leq 3$  as shown in Fig. 5.1 we get a mesh on  $\Omega$ :  $\{\Omega_{i,j}, \text{ for } i = 1,$

$1 \leq j \leq n+1, 2 \leq i \leq 3, j = 2, \Omega_{i,j}^k \text{ for } 1 \leq k \leq 3, 2 \leq i \leq 3, 2 < j \leq n+1\}$ .

We denote this mesh by  $\Omega_\sigma^n$  and call it a geometric mesh.

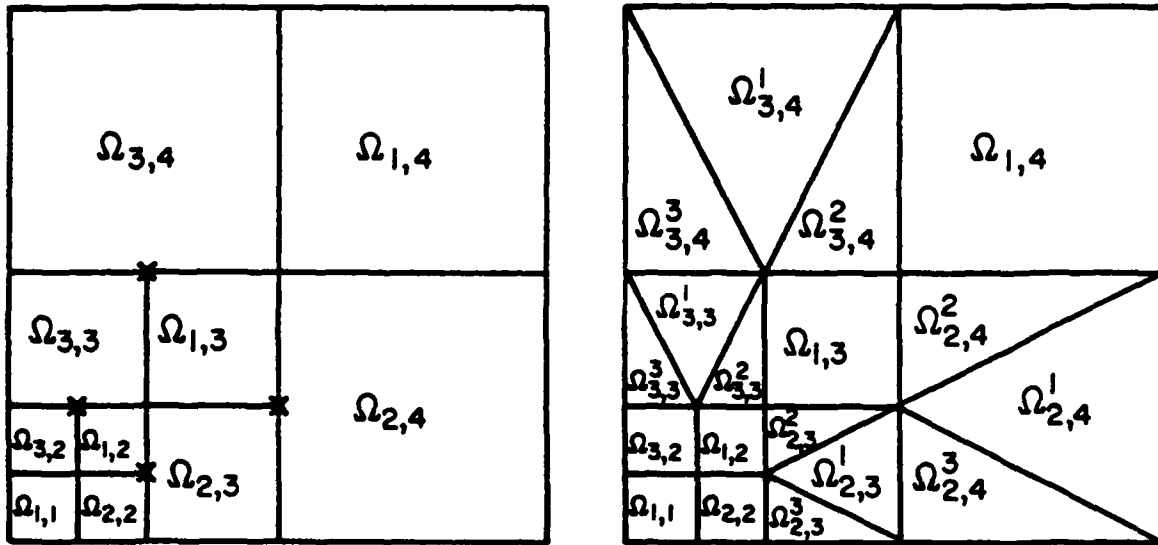


Figure 5.1

Geometric Mesh on a Square

Let  $S_{\sigma}^{\underline{P}, \underline{Q}, 1}(\Omega_{\sigma}^n) \subset H^1(\Omega)$  be subspace of functions which restricted to  $\Omega_{i,j}^k$ ,  $1 \leq k \leq 3$ ,  $2 \leq i \leq 3$ ,  $2 < j \leq n+1$  (resp.  $\Omega_{i,j}$ ,  $i=1$ ,  $1 \leq j \leq n+1$ , or  $1 \leq i \leq 3$ ,  $j=2$ ) are polynomials of degree  $\leq p_{i,j}^k$  (resp.  $p_{i,j}$ ) in  $x$  and of degree  $\leq q_{i,j}^k$  (resp.  $q_{i,j}$ ) in  $y$ . By  $\underline{P}$  and  $\underline{Q}$  we denote the degree vectors  $\underline{P} = (p_{i,j}^k) = (p_{1,1}, p_{1,2}, p_{2,2}, \dots, p_{3,n+1}^3)$  and  $\underline{Q} = (q_{i,j}^k) = (q_{1,1}, q_{1,2}, q_{2,2}, \dots, q_{3,n+1}^3)$ .

**Lemma 5.1.** Assume that  $\Omega = (0,1) \times (1,0)$ ,  $u \in B_{\beta,d}^2(\Omega)$  and  $\phi_{\beta} = r^{\beta}$ . Then there exists  $\phi(x,y) \in S_{\sigma}^{\underline{P}, \underline{Q}, 1}(\Omega_{\sigma}^n)$ ,  $0 < \sigma < 1$ ,  $p_{1,1} = q_{1,1} = 1$ ,  $p_{i,j}^k \geq 2$ ,  $q_{i,j}^k = q_{i,j} \geq 2$ ,  $1 \leq k \leq 3$ ,  $j \geq 2$ , and  $p_{i,j}$ ,  $q_{i,j}$  are non-decreasing with  $j$  such that

$$\begin{aligned}
(5.1) \quad \|u - \phi\|_{H^1(\Omega)}^2 &\leq c \{h_{1,1}^{2(1-\beta)} \|u\|_{H^2(\Omega_{1,1})}^2 \\
&+ (\sum_{1 \leq i \leq 3, 2 \leq j \leq n+1} x_{j-1}^{2(1-\beta)} \frac{(p_{i,j} - s_{i,j})!}{(p_{i,j} + s_{i,j} - 2)!} \left(\frac{\lambda_{i,j}}{2}\right)^{2s_{i,j}} \|u\|_{H_\beta^{s_{i,j}+3,2}(\Omega_{i,j})}^2 \\
&+ x_{j-1}^{2(1-\beta)} \frac{(q_{i,j} - t_{i,j})!}{(q_{i,j} + t_{i,j} - 2)!} \left(\frac{v_{i,j}}{2}\right)^{2t_{i,j}} \|u\|_{H_\beta^{t_{i,j}+3,2}(\Omega_{i,j})}^2) \}
\end{aligned}$$

where  $\phi_{1,1} = \phi|_{\Omega_{1,1}}$  is bilinear interpolation of  $u$  at vertices of  $\Omega_{1,1}$ ,

$s_{i,j}$  and  $t_{i,j}$  are real numbers,  $1 \leq s_{i,j} \leq p_{i,j}$ ,  $1 \leq t_{i,j} \leq q_{i,j}$ ,  $\lambda_{1,j} = v_{1,j} = (1-\sigma)/(\sqrt{2}\sigma)$ ,  $\lambda_{2,j} = v_{3,j} = (1-\sigma)/\sigma$ ,  $v_{2,j} = \lambda_{3,j} = 1$  for  $2 \leq i \leq n+1$ , and  $(p_{i,j} - s_{i,j})! = \Gamma(p_{i,j} - s_{i,j} + 1)$ , etc.

Proof. Applying Theorem 4.1 on each  $\Omega_{i,j}$ ,  $1 \leq i \leq 3$ ,  $2 \leq j \leq n+1$ , there exists a polynomial  $\phi_{i,j}$  with degree  $p_{i,j}$  in  $x$  and  $q_{i,j}$  in  $y$  such that for  $0 \leq m \leq 2$

$$\begin{aligned}
(5.2) \quad \|D^m(u - \phi_{i,j})\|_{H^0(\Omega_{i,j})}^2 \\
\leq c x_{j-1}^{2(2-m-\beta)} \{ \frac{(p_{i,j} - s_{i,j})!}{(p_{i,j} + s_{i,j} + 2 - 2m)!} \left(\frac{\lambda_{i,j}}{2}\right)^{2s_{i,j}} \|u\|_{H_\beta^{s_{i,j}+3,2}(\Omega_{i,j})}^2 \\
+ \frac{(q_{i,j} - t_{i,j})!}{(q_{i,j} + t_{i,j} + 2 - 2m)!} \left(\frac{v_{i,j}}{2}\right)^{2t_{i,j}} \|u\|_{H_\beta^{t_{i,j}+3,2}(\Omega_{i,j})}^2 \}.
\end{aligned}$$

On the regular nodal point of  $\Omega_{\sigma}^{*n}$  the polynomial  $\phi_{i,j}(x,y)$  coincides with  $u(x,y)$ . We need to adjust  $\phi_{i,j}$  on  $\Omega_{i,j}^k$  for the coincidence on the irregular nodal point.

We show the treatment on  $\Omega_{2,j}$  ( $j > 2$ ). We denote  $\phi_{2,j}(x,y)$  on  $\Omega_{2,j}^k$  by  $\phi_{2,j}^k(x,y)$ ,  $1 \leq k \leq 3$ , and let  $v_{2,j}^k$  be linear function on  $\Omega_{2,j}^k$  such that

$$v_{2,j}^k(x_{j-1}, y_{j-2}) = u(x_{j-1}, y_{j-2}) - \phi_{2,j}^k(x_{j-2}, y_{j-2}),$$

$$v_{2,j}^k = 0 \text{ on the other two vertices of } \Omega_{2,j}^k.$$

Then setting

$$\tilde{\phi}_{2,j}^k = \phi_{2,j}^k + v_{2,j}^k,$$

$$\text{on } \Omega_{2,j}^k, \quad 1 \leq k \leq 3$$

we have

$$u = \tilde{\phi}_{2,j}^k \text{ on all vertices of } \Omega_{2,j}^k,$$

$$1 \leq k \leq 3.$$

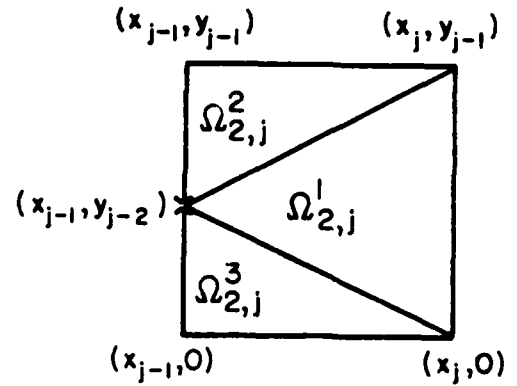


Figure 5.2

Partition of a Nondiagonal Element

Applying Lemma 3.5 we have for  $i \geq 2, j > 2$

$$(5.3) \quad \|u - \tilde{\phi}_{i,j}^k\|_{H^2(\Omega_{i,j}^k)}^2 \leq \|u - \phi_{i,j}^k - v_{i,j}^k\|_{H^2(\Omega_{i,j}^k)}^2 \leq C |u - \phi_{i,j}^k|_{H^2(\Omega_{i,j}^k)}^2,$$

$$(5.4) \quad \|u - \tilde{\phi}_{i,j}^k\|_{H^1(\Omega_{i,j}^k)}^2 \leq Ch_{i,j}^2 |u - \phi_{i,j}^k|_{H^2(\Omega_{i,j}^k)}^2.$$

Functions  $\phi_{i,j}$  ( $i = 1$ , or  $j < 2$ ) and  $\tilde{\phi}_{i,j}^k$  ( $i \geq 2$ ,  $j > 2$ ) should be further adjusted for the continuity on the common edge of two triangles or rectangles.

There are six basic cases for  $i = 2$  shown in Fig. 5.3, and another similar six cases for  $i = 3$ , which we have to treat.

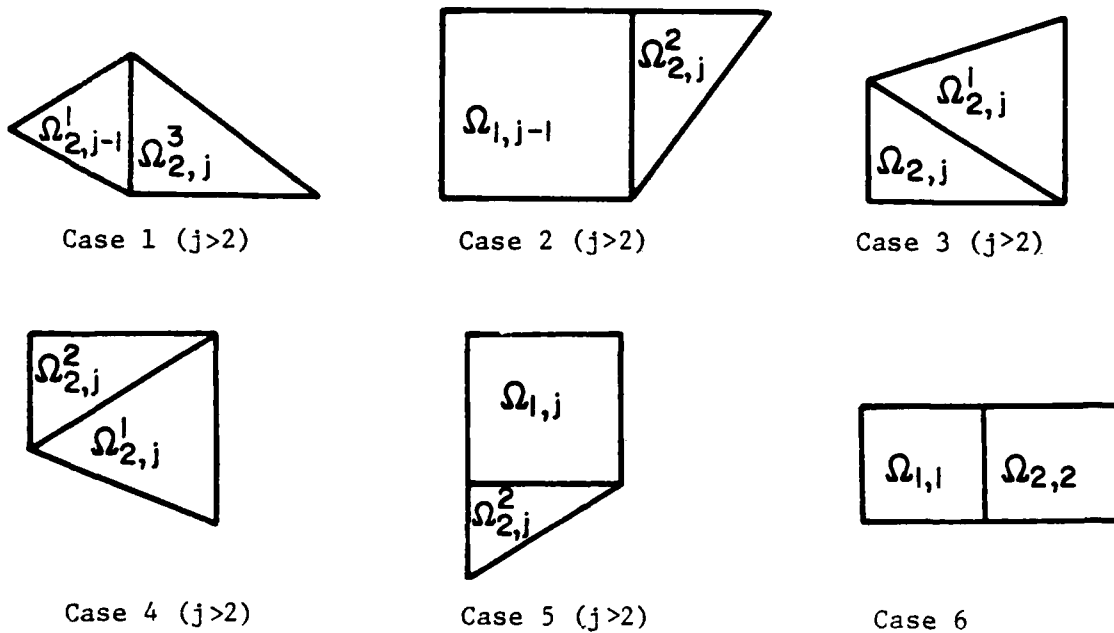


Figure 5.3

#### Scheme of Adjacent Elements

We show the treatment of Case 2, Case 3 and Case 6. The other cases are similar. In Case 3  $\tilde{\phi}_{2,j}^1$  and  $\tilde{\phi}_{2,j}^3$  coincide with  $u$  at endpoints of common side  $\gamma$  of  $\Omega_{2,j}^1$  and  $\Omega_{2,j}^3$ . Let  $w = (\tilde{\phi}_{2,j}^1 - \tilde{\phi}_{2,j}^3)$ .

Since  $\tilde{\phi}_{2,j}^1 - \tilde{\phi}_{2,j}^3 = v_{2,j}^1 - v_{2,j}^3$  is a linear function,  $w$  vanishes on  $\gamma$ .

Hence the piecewise polynomial is already continuous on  $\Omega_{2,j}^1 \cup \Omega_{2,j}^3$

( $j > 2$ ). In Case 2, the functions  $\phi_{1,j-1}$  and  $\tilde{\phi}_{2,j}^2$  coincide with  $u$  at the vertices of  $\Omega_{1,j-1}$  and  $\Omega_{2,j}^2$ ,  $w = (\phi_{1,j-1} - \tilde{\phi}_{2,j}^2)|_{\gamma}$  vanishes at the endpoints of common side  $\gamma$  of  $\Omega_{1,j-1}$  and  $\Omega_{2,j}^2$ . By assumption  $w$  is a polynomial of degree  $\leq q_{2,j}$  in  $y$ . By Lemma 3.3 there exists a polynomial  $\hat{w}$  of degree  $q_{2,j}$  in  $y$  and degree 1 in  $x$  such that for  $0 \leq y \leq y_{j-1}$ ,  $\hat{w}(x_{j-2}, y) = w(y)$ , it vanishes on other sides of  $\Omega_{2,j}^2$ , and

$$(5.5) \quad \|\hat{w}\|_{H^1(\Omega_{2,j}^2)}^2 \leq Ch_{2,j} \|w\|_{H^1(\gamma)}^2 \\ \leq Ch_{2,j} \left( \|\tilde{\phi}_{2,j}^2 - u\|_{H^1(\gamma)}^2 + \|\phi_{1,j-1} - u\|_{H^1(\gamma)}^2 \right)$$

by Lemma 3.1

$$\leq C \{ h_{2,j}^2 \|\tilde{\phi}_{2,j}^2 - u\|_{H^2(\Omega_{2,j}^2)}^2 + h_{2,j} h_{2,j-1} \|\phi_{1,j-1} - u\|_{H^2(\Omega_{1,j-1})}^2 \} \\ \leq \tilde{C} \{ h_{2,j}^2 \|\tilde{\phi}_{2,j}^2 - u\|_{H^2(\Omega_{2,j}^2)}^2 + h_{2,j-1}^2 \|\phi_{1,j-1} - u\|_{H^2(\Omega_{2,j-1})}^2 \}.$$

Setting

$$\begin{cases} \psi_{2,j}^2 = \tilde{\phi}_{2,j}^2 - \hat{w} & \text{in } \Omega_{2,j}^2, \\ \psi_{1,j-1} = \phi_{1,j-1} & \text{in } \Omega_{1,j-1} \end{cases}$$



we have

$$\begin{aligned}
 (5.6) \quad \|u - \psi_{2,j}^2\|_{H^1(\Omega_{2,j}^2)}^2 &\leq 2\|u - \tilde{\phi}_{2,j}^2\|_{H^1(\Omega_{2,j}^2)}^2 + \|\hat{w}\|_{H^1(\Omega_{2,j}^2)}^2 \\
 &\leq \{\|u - \tilde{\phi}_{2,j}^2\|_{H^1(\Omega_{2,j}^2)}^2 + h_{2,j}^2 \|u - \tilde{\phi}_{2,j}^2\|_{H^2(\Omega_{2,j}^2)}^2 + h_{1,j-1}^2 \|u - \tilde{\phi}_{1,j-1}\|_{H^2(\Omega_{1,j-1}^2)}^2\}
 \end{aligned}$$

by (5.3) and (5.4)

$$\leq C\{h_{2,j}^2 \|u - \phi_{2,j}\|_{H^2(\Omega_{2,j}^k)}^2 + h_{1,j-1}^2 \|u - \phi_{1,j-1}\|_{H^2(\Omega_{1,j-1}^1)}^2\}$$

and

$$(5.7) \quad \|u - \psi_{1,j-1}\|_{H^1(\Omega_{1,j-1}^1)}^2 = \|u - \phi_{1,j-1}\|_{H^1(\Omega_{1,j-1}^1)}^2.$$

In Case 6 ( $j = 2$ )  $\phi_{1,1}$  is the bilinear interpolation of  $u$  at vertices of  $\Omega_{1,1}$ . The common edge of  $\Omega_{1,1}$  and  $\Omega_{2,2}$  is separated from the origin and  $w = (\phi_{2,2} - \phi_{1,1})|_{\gamma}$  vanishes at endpoint of  $\gamma$ . By Lemma 3.3 there exists polynomial of degree 2 in  $y$  and degree 1 in  $x$  such that  $\hat{w}|_{\gamma} = w$  and  $\hat{w}$  vanishes on the other edge of  $\Omega_{2,2}$  and

$$\|\hat{w}\|_{H^1(\Omega_{2,2})}^2 \leq Ch_{2,2}^2 \|w\|_{H^1(\gamma)}^2$$

$$\|\hat{w}\|_{H^1(\Omega_{2,2})}^2 \leq Ch_{2,2} \|w\|_{H^1(\gamma)}^2$$

by Lemma 3.1 and Lemma 3.2

$$\begin{aligned} &\leq Ch_{2,2} (h_{2,2} \|w-\phi_{2,2}\|_{H^2(\Omega_{2,2})}^2 + h_{1,1}^{1-2\beta} \|u-\phi_{1,1}\|_{H_\beta^{2,2}(\Omega_{1,1})}^2) \\ &\leq C (h_{2,2}^2 \|w-\phi_{2,2}\|_{H^2(\Omega_{2,2})}^2 + h_{1,1}^{2(1-\beta)} \|u-\phi_{1,1}\|_{H_\beta^{2,2}(\Omega_{1,1})}^2). \end{aligned}$$

Letting

$$\begin{cases} \psi_{2,2} = \phi_{2,2} - \hat{w} & \text{in } \Omega_{2,2}, \\ \psi_{1,1} = \phi_{1,1} & \text{in } \Omega_{1,1}, \end{cases}$$

we have

$$\begin{aligned} (5.8) \quad &\|u-\psi_{2,2}\|_{H^1(\Omega_{2,2})}^2 \leq \|u-\phi_{2,2}\|_{H^1(\Omega_{2,2})}^2 + \|\hat{w}\|_{H^1(\Omega_{2,2})}^2 \\ &\leq C \{ \|u-\phi_{2,2}\|_{H^1(\Omega_{2,2})}^2 + h_{2,2}^2 \|u-\phi_{2,2}\|_{H^2(\Omega_{2,2})}^2 + h_{1,1}^{2(1-\beta)} \|u-\phi_{1,1}\|_{H_\beta^{2,2}(\Omega_{1,1})}^2 \} \end{aligned}$$

and by Lemma 3.6

$$(5.9) \quad \|u-\psi_{1,1}\|_{H^1(\Omega_{1,1})}^2 \leq Ch_{1,1}^{2(1-\beta)} \|u-\phi_{1,1}\|_{H_\beta^{2,2}(\Omega_{1,1})}^2.$$

After treating all the cases we obtain modified  $\psi_{i,j}^k$ ,  $i > 1$ ,  $j > 2$  and modified  $\psi_{i,2}$  ( $i = 1, 2$ ). Define

$$\psi = \begin{cases} \psi_{i,j}^k & \text{on } \Omega_{i,j}^k \text{ for } 1 \leq k \leq 3, 2 \leq i \leq 3, 3 \leq j \leq n+1, \\ \psi_{i,j} & \text{on } \Omega_{i,j} \text{ for } i = 1, 1 \leq j \leq n+2 \text{ and } 1 \leq i \leq 3, j = 2. \end{cases}$$

Hence  $\psi \in S_{\sigma}^{\underline{P}, \underline{Q}, 1}(\Omega_{\sigma}^n)$  and

$$(5.10) \quad \|u - \psi\|_{H^1(\Omega)}^2 \leq C \{ h_{1,1}^{2(1-\beta)} \|u - \phi_{1,1}\|_{H_{\beta}^{2,2}(\Omega_{1,1})}^2 + \sum_{1 \leq i \leq 3, 1 < j \leq n+1} (\|u - \phi_{i,j}\|_{H^1(\Omega_{i,j})}^2 + h_{i,j}^2 \|u - \phi_{i,j}\|_{H^2(\Omega_{i,j})}^2) \}.$$

(3.5), (5.2) and (5.10) yield (5.1) □

**Theorem 5.1.** Let  $\Omega = (0,1) \times (0,1)$  and  $\Omega_{\sigma}^n$  be the geometric mesh on  $\Omega$  shown in Fig. 5.1. If  $u \in B_{\beta,d}^2(\Omega)$  with  $0 < \beta < 1$  and  $\phi_{\beta} = r^{\beta}$ , then for any  $0 < \sigma < 1$  there exists a piecewise polynomial  $\psi(x,y) \in S_{\sigma}^{\underline{P}, \underline{Q}, 1}(\Omega_{\sigma}^n)$ , with degree vectors  $\underline{P}$  and  $\underline{Q}$  in which  $p_{1,1} = q_{1,1} = 1$ ,  $p_{i,j}^k = p_{i,j} = p_j$ ,  $q_{i,j}^k = q_{i,j} = p_j$  for  $1 \leq k \leq 3$ ,  $1 \leq i \leq 3$ ,  $1 < j \leq n+1$ , and  $p_j = \max(2, [j\mu])$ , and  $\mu$  satisfying (5.15) such that

$$(5.11) \quad \|u - \psi\|_{H^1(\Omega)} \leq C e^{-bN^{1/3}}.$$

The constants  $C$  and  $b$  are independent of  $\underline{P}$ ,  $\underline{Q}$  and the number  $N$  of degrees of freedom of the space  $S_{\sigma}^{\underline{P}, \underline{Q}, 1}(\Omega_{\sigma}^n)$ .

Proof. Applying Lemma 5.1 there is  $\psi \in S^{\underline{p}, \underline{q}, 1}(\Omega_\sigma^n)$  satisfying

$$\begin{aligned} \|u - \psi\|_{H^1(\Omega)}^2 &\leq C \left\{ \sum_{\substack{1 \leq j \leq n+1 \\ 1 \leq i \leq 3}} x_{j-1}^{2(1-\beta)} \left( \frac{(p_{i,j} - s_{i,j})!}{(p_{i,j} + s_{i,j} - 2)!} \left( \frac{\lambda_{i,j}}{2} \right)^{2s_{i,j}} \|u\|_{H_\beta^{s_{i,j}+3,2}(\Omega_{i,j})}^2 \right. \right. \\ &\quad \left. \left. + \frac{(q_{i,j} - t_{i,j})!}{(q_{i,j} + t_{i,j} - 2)!} \left( \frac{\nu_{i,j}}{2} \right)^{2t_{i,j}} \|u\|_{H_\beta^{t_{i,j}+3,2}(\Omega_{i,j})}^2 \right) + \sigma^{2(1-\beta)(n+1)} \right\}. \end{aligned}$$

Letting  $\rho = \max(1, \frac{1-\sigma}{\sigma})$  we have using (2.3)

$$\begin{aligned} \|u - \psi\|_{H^1(\Omega)}^2 &\leq C \left\{ \sum_{1 \leq j \leq n+1} x_{j-1}^{2(1-\beta)} \left( \frac{(p_{i,j} - s_{i,j})!}{(p_{i,j} + s_{i,j} - 2)!} \left( \frac{\rho d}{2} \right)^{2s_{i,j}} ((s_{i,j} + 1)!)^2 \right. \right. \\ &\quad \left. \left. + \frac{(q_{i,j} - t_{i,j})!}{(q_{i,j} + t_{i,j} - 2)!} \left( \frac{\rho d}{2} \right)^{2t_{i,j}} ((t_{i,j} + 1)!)^2 + \sigma^{2(1-\beta)(n+1)} \right\}. \end{aligned}$$

Let  $s_{i,j} = \alpha_j p_{i,j} = \alpha_j p_j$ ,  $t_{i,j} = \bar{\alpha}_j q_{i,j} = \bar{\alpha}_j q_j$ ,  $0 < \alpha_j, \bar{\alpha}_j \leq 1$  with  $\alpha_j, \bar{\alpha}_j$  being determined later. Then

$$\begin{aligned} (5.12) \quad \|u - \psi\|_{H^1(\Omega)}^2 &\leq C \left\{ \sum_{1 \leq j \leq n+1} x_{j-1}^{2(1-\beta)} \left( |F(\rho d, \alpha_j)|^{p_j \cdot p_j^6} \right. \right. \\ &\quad \left. \left. + |F(\rho d, \bar{\alpha}_j)|^{q_j \cdot q_j^6} \right) + \sigma^{2(1-\beta)(n+1)} \right\} \end{aligned}$$

where  $F(d_1, \alpha) = \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}} \left( \frac{\alpha d_1}{2} \right)^{2\alpha}$ . Function  $F(d_1, \alpha)$  is defined on  $[0, \infty) \times [0, 1]$  and  $F(\rho d, \frac{2}{\rho d}) < 1$ .

Hence

$$(5.13) \quad F_{\min} = \min_{\alpha \in [0, 1]} F(\rho d, \alpha) = F(\rho d, \alpha_{\min}) \leq F(\rho d, \frac{2}{\rho d}) < 1, \quad \alpha_{\min} = 2/\sqrt{4 + \rho^2 d^2}.$$

Taking  $p_j = q_j$ ,  $\alpha_j = \bar{\alpha}_j = \max(\frac{1}{p_j}, \alpha_{\min})$  we have by (5.12)

$$(5.14) \quad \|u-\psi\|_{H^1(\Omega)}^2$$

$$\leq c \left( \sigma^{2(n+1)(1-\beta)} + \sum_{1 \leq j \leq j_0} x_{j-1}^{2(1-\beta)} |F(\rho_d, \frac{1}{p_j})|^{p_j} p_j^6 \right. \\ \left. + \sum_{1+j_0 < j \leq n+1} x_{j-1}^{2(1-\beta)} |F_{\min}|^{p_j} p_j^6 \right)$$

where  $p_{j_0} = \left[ \frac{1}{\alpha_{\min}} \right] + 1$ ,  $[a]$  means the integral part of  $a$ . We choose  $p_j = \max(2, [j\mu])$  for  $j \geq 2$  and

$$(5.15) \quad \mu > \frac{2(1-\beta)\ln\sigma}{\ln F_{\min}}.$$

Since  $j_0$  is bounded

$$j_0 \leq \frac{1}{\mu \alpha_{\min}} + \frac{2}{\mu},$$

and

$$(5.16) \quad \|u-\psi\|_{H^1(\Omega)}^2 \leq c \sigma^{2(n+1)(1-\beta)} \\ \left\{ 1 + \sum_{2 \leq j \leq j_0} \frac{|F_{\min}|^{p_j} p_j^6}{\sigma^{2(1-\beta)j}} \max_{2 \leq j \leq j_0} \left| \frac{F(\rho_d, \frac{1}{p_j})^{p_j}}{F_{\min}} \right| + \sum_{j_0+1 \leq j \leq n+1} \frac{|F_{\min}|^{p_j} p_j^6}{\sigma^{2(1-\beta)j}} \right\} \\ \leq \tilde{c} \sigma^{2(n+2)(1-\beta)}$$

with  $\tilde{c} > 0$  independent of  $n$ . Let  $\bar{\mu} = \max(1, \mu)$ . The number of degrees of freedom  $N$  is

$$N \leq 3 + 3(2\bar{\mu}+1)^2 + 7 \sum_{j=3}^{n+1} (\bar{\mu}j+1)^2 \leq \frac{7}{3} \bar{\mu}^2 (n+2+\frac{1}{\bar{\mu}})^3.$$

Hence

$$\|u-\psi\|_{H^1(\Omega)}^2 \leq C e^{-2(1-\beta) \ln \frac{1}{\sigma} \sqrt{\frac{3}{7\bar{\mu}^2}} N^{1/3}}$$

which yields (5.11) with  $b = (1-\beta) \left( \frac{3}{7\bar{\mu}^2} \right)^{1/3} \ln \frac{1}{\sigma}$  and  $C$  independent of  $N$ .  $\square$

Remark 1. In practice  $p_j$  and  $q_j$  are often selected uniformly,  $p_j = q_j = p = [\mu(n+1)]$  for some  $\mu > 0$ . Then (5.11) holds and (5.15) is not needed. From (5.16) we see that for  $p_j = p$

$$\begin{aligned} (5.17) \quad \|u-\psi\|_{H^1(\Omega)}^2 &\leq C \sigma^{2(n+2)(1-\beta)} \{1 + |F_{\min}|^p p^6 \sum_{1 \leq j \leq n+1} (\frac{1}{\sigma})^{2(1-\beta)j}\} \\ &\leq C \{ \sigma^{2(n+2)(1-\beta)} + |F_{\min}|^p p^6 \} \\ &\leq C \{ \sigma^{2(1-\beta)\mu \frac{2}{3} (\frac{N}{7})} + |F_{\min}|^{\mu \frac{1}{3} (\frac{N}{7})} p^6 (\frac{\mu N}{7})^2 \} \\ &\leq C e^{-bN^{\frac{1}{3}}} \end{aligned}$$

with  $b = \frac{(1-\beta)}{3\sqrt{7}} \min(\mu^{\frac{2}{3}} \ln \frac{1}{\sigma}, \mu^{\frac{1}{3}} \ln \frac{1}{F_{\min}})$  and  $C$  independent of  $N$ . In

(5.17) the first term is the error on the element containing singularity, the second is the error on the rest of the elements.

Let us discuss now the optimal choice of  $\mu$ . First we drop the term  $p^6$  in (5.17) since  $|F_{\min}|^p p^6 \leq CF^p$  for some  $0 < F < F_{\min} < 1$ , and replace  $\sqrt[3]{\frac{N}{7}}$  and  $\mu^{1/3}$  by  $N_0$  and  $v$  respectively. Consider the function  $g(v) = g_1(v) + g_2(v)$ ,  $g_1(v) = \sigma^{2(1-\beta)N_0} v^{-2}$  and  $g_2(v) = F^v N_0$ , with  $0 < F, \sigma < 1$ . It can be readily seen that  $g_1(v)$  and  $g_2(v)$  are increasing, respectively decreasing functions on  $[0, \infty)$ , and  $g_1(v) = g_2(v)$  for  $v = v_0 = \frac{2(1-\beta)\ln \sigma}{\ln F}^{1/3}$  with  $v_0$  independent of  $N_0$ .  $g'_1(v)$  and  $g'_2(v)$  intersect each other at least once and three times at most, (see Fig. 5.4). Therefore there are  $v_i$ ,  $i = 1, 2, 3$  (resp.  $i = 1, 2$  and  $i = 1$ ) such that

$$(5.18) \quad v_i^2 = \frac{4(1-\beta) \sigma^{2(1-\beta)N_0} v_i^{-2} \ln \sigma}{F^{N_0} v_i \ln F}$$

since  $g'(v_0) > 0$ ,  $g'(0) < 0$ , and  $v_1 < v_0$ .

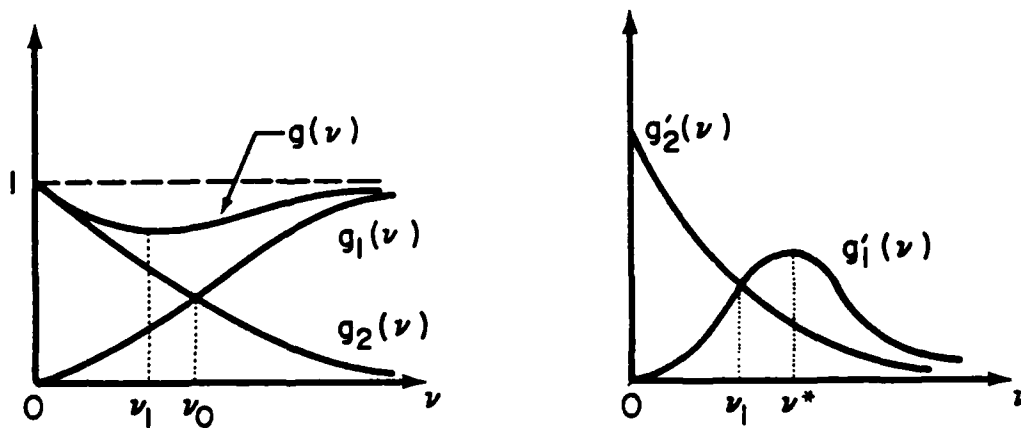


Figure 5.4

Determination of the Optimal Value of  $v$

Now we can speak of the optimal value  $\mu = v_1^3$  (resp.  $v_3^3$ ) in the sense of minimizing  $g(v)$  at  $v_1$  (resp.  $v_3$ ). The optimal value of  $\mu$  depends on  $\sigma$ ,  $F$ ,  $\beta$ , and  $N$ .

For the asymptotic analysis we write

$$g(v) = \sigma^{2(1-\beta)N_0 v^2 + \sigma \frac{\ln F}{\ln \sigma} v N_0}.$$

If  $0 < v < v_0$ ,  $g(v) \approx \sigma^{\frac{\ln F}{\ln \sigma} N_0}$  for large  $N_0$ , which is decreasing with  $v$ .

If  $v > v_0$ ,  $g(v) \approx \sigma^{2(1-\beta)N_0 v^2}$ , which increases with  $v$ . Hence  $v_0$  is the asymptotic value of the optimal  $v$ . For large  $N$  equation (5.18) has only one solution,  $v_1$  and  $v_3$  will approach to  $v_0$  from left and right respectively.

Remark 2.  $\alpha_{\min}$  and  $F_{\min}$  are the function of  $d_1 = \rho d$ , and  $F_{\min} = F(d_1, \alpha_{\min}(d_1))$ .

We see that

$$F_{\min}(d_1) = (F_{d_1} + \alpha'_{\min}(d_1) F_{\alpha})_{\alpha=\alpha_{\min}} > 0,$$

and  $F_{\min}$  is increasing function of  $d_1$ . If  $\rho$  and  $d$  small, we can choose small  $\mu$ ; hence we can reduce the number of degrees of freedom of  $S_{P,Q,1}^{n}(\Omega_{\sigma}^n)$ . The value  $d$  is related to the function  $u(x,y)$  and  $\rho$  is related to the mesh.

Remark 3. The mesh  $\Omega_{\sigma}^{*n}$  with the irregular points can be used analogously.

The results are then quite the same as for the triangular mesh we analyzed above.

Remark 4. The technique we used for the adjustment of two adjacent elements can be directly applied to satisfy homogeneous (or polynomial) boundary condition.



So far the singularity of the solution occurred in one vertex only.

The general case can be easily treated.

Let  $u \in B_{\beta}^2(\Omega)$  with  $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ ,  $0 < \beta_i < 1$ ,  $1 \leq i \leq 4$  and  $\Omega = (0,2) \times (0,2)$ . Divide  $\Omega$  into four subdomains  ${}^m\Omega$ ,  $1 \leq m \leq 4$  such that on each  ${}^m\Omega$  the function  $u$  has only one singular point, and apply Theorem 5.1 on  ${}^m\Omega$ ,  $1 \leq m \leq 4$ . Let  $\Omega_{\sigma}^n$  be the union of the geometric meshes  $\Omega_{\sigma}^n$  on  ${}^m\Omega$ ,  $1 \leq m \leq 4$ . Let  $\underline{P} = ({}^1\underline{P}, {}^2\underline{P}, {}^3\underline{P}, {}^4\underline{P})$  and  $\underline{Q} = ({}^1\underline{Q}, {}^2\underline{Q}, {}^3\underline{Q}, {}^4\underline{Q})$  be the union of  ${}^m\underline{P}$  and  ${}^m\underline{Q}$ ,  $1 \leq m \leq 4$ , each  ${}^m\underline{P}$  and  ${}^m\underline{Q}$  be defined as before. Thus for the mesh  $\Omega_{\sigma}^n$  and degree vectors  $\underline{P}$  and  $\underline{Q}$  (see Fig. 5.5) we have

Theorem 5.2. Let  $\Omega = (0,2) \times (0,2)$ ,  $\Omega_{\sigma}^n$ ,  $\underline{P}$ ,  $\underline{Q}$  be defined as above.

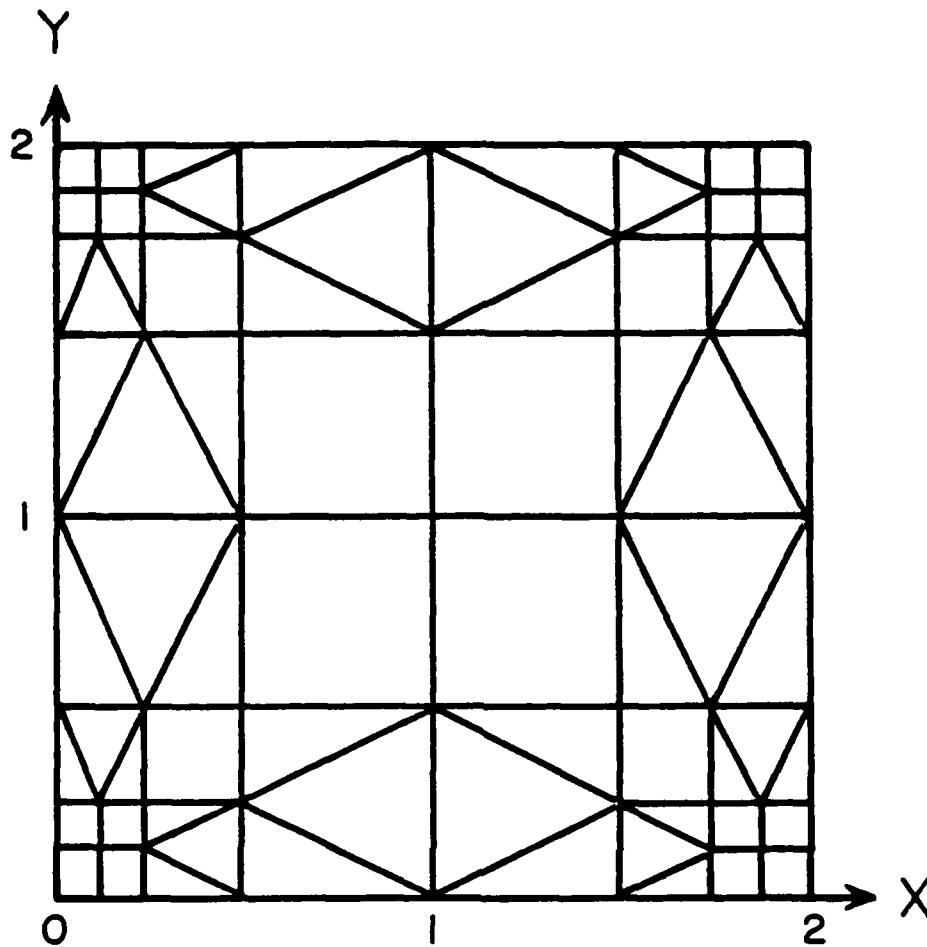
If  $u \in B_{\beta,d}^2(\Omega)$  with  $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ ,  $0 < \beta_i < 1$ ,  $1 \leq i \leq 4$ , then for any  $0 < \sigma < 1$ , there exists a piecewise polynomial  $\psi(x,y) \in S^{\underline{P},\underline{Q},1}(\Omega_{\sigma}^n)$  such that

$$\|u - \psi\|_{H^1(\Omega)}^2 \leq C e^{-bN^{1/3}}$$

with constant  $C$  independent of  $N$  and  $b = 2(1-\bar{\beta}) \ln \frac{1}{\sigma} \left( \frac{3}{28\bar{\mu}^2} \right)^{1/3}$  where

$\bar{\beta} = \max_{1 \leq i \leq 4} \beta_i$ ,  $\bar{\mu} = \max(1, \mu)$  and  $\mu$  satisfies (5.15).  $\square$

We shall now briefly address the geometric mesh on a parallelogram domain  $\Omega$ , which is the image of geometric mesh on square under a linear mapping. Let  $M$  be a mapping of standard square  $S = (0,1) \times (0,1)$  onto  $\Omega$ :



$$\sigma = 0.5$$

$$n = 3$$

$$\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$$

$$0 < \beta_i < 1 \quad 1 \leq i \leq 4$$

Figure 5.5

Geometric Mesh on a Square

$$(5.18) \quad \begin{cases} x = a\xi + b\eta \cos \omega \\ y = b\eta \sin \omega \end{cases}$$

with Jacobian =  $a b \sin \omega$ . The inverse of  $M$  is given by

$$(5.19) \quad \begin{cases} \xi = \psi_1(x, y) = (x - y \cot \omega)/a \\ \eta = \psi_2(x, y) = y \sin \omega / b \end{cases}$$

where  $\omega$  is the interior angle of  $\Omega$  and  $a, b$  are the lengths of the sides of  $\Omega$  shown in Fig. 5.6. Let  $\Omega_G^n = M(S_G^n)$ , where  $S_G^n$  is the geometric mesh of square  $S$  (see Fig. 5.2).

Let  $S^{P,Q,1}(\Omega_G^n)$  be the subspace of continuous and piecewise polynomials  $\phi(x, y) \in H^1(\Omega)$  such that

$$\phi|_{\Omega_{i,j}^k} = \sum_{0 \leq \ell \leq p_{i,j}} \sum_{0 \leq m \leq q_{i,j}^k} C_{\ell,m} \psi_1^\ell \psi_2^m \quad \text{for } 1 \leq k \leq 3, 2 \leq i \leq 3, 2 \leq j \leq n+1,$$

and

$$\phi|_{\Omega_{i,j}} = \sum_{0 \leq \ell \leq p_{i,j}} \sum_{0 \leq m \leq q_{i,j}} C_{\ell,m} \psi_1^\ell \psi_2^m \quad \text{for } i = 1, 1 \leq j \leq n+1 \text{ and } i = 2, 3, \\ j = 2.$$

The number of degrees of freedom of  $S^{P,Q,1}(S_G^n)$  is defined as the number

of degrees of freedom of  $S_{\underline{P}, \underline{Q}, 1}^n(\Omega_\sigma^n)$  and denoted by  $N$ . We have the following theorem

**Theorem 5.3.** Let  $\Omega$  be a parallelogram shown in Fig. 5.6 with interior angle  $\omega$ ,  $0 < \omega < \frac{\pi}{2}$ . Then for any  $u \in B_{\beta, d}^2(\Omega)$  with  $\phi_\beta = r^\beta$ ,  $0 < \beta < 1$ , and  $0 < \sigma < 1$ , there exists  $\psi(x, y) \in S_{\underline{P}, \underline{Q}, 1}^n(\Omega_\sigma^n)$  with the degree vector  $\underline{P}$  and  $\underline{Q}$  in which  $p_{1,1} = q_{1,1} = 1$ ,  $p_{i,j} = q_{i,j} = p_{i,j}^k = q_{i,j}^k = \max(2, [j\mu])$ ,  $j \geq 2$  and  $\mu$  satisfying (5.15), with  $\bar{d}$  replaced by  $\bar{d} = d/\sqrt{2} \max(a, b) / \min(a, b)$  such that

$$\|u - \psi\|_{H^1(\Omega)}^2 \leq C e^{-bN^{1/3}}$$

with  $b = 2(1-\beta)^3 \left(\frac{3}{7\bar{\mu}^2}\right)^{1/3} \ln \frac{1}{\sigma}$ ,  $\bar{\mu} = \max(1, \mu)$  and constant  $C$  independent of  $N$ .  $\square$

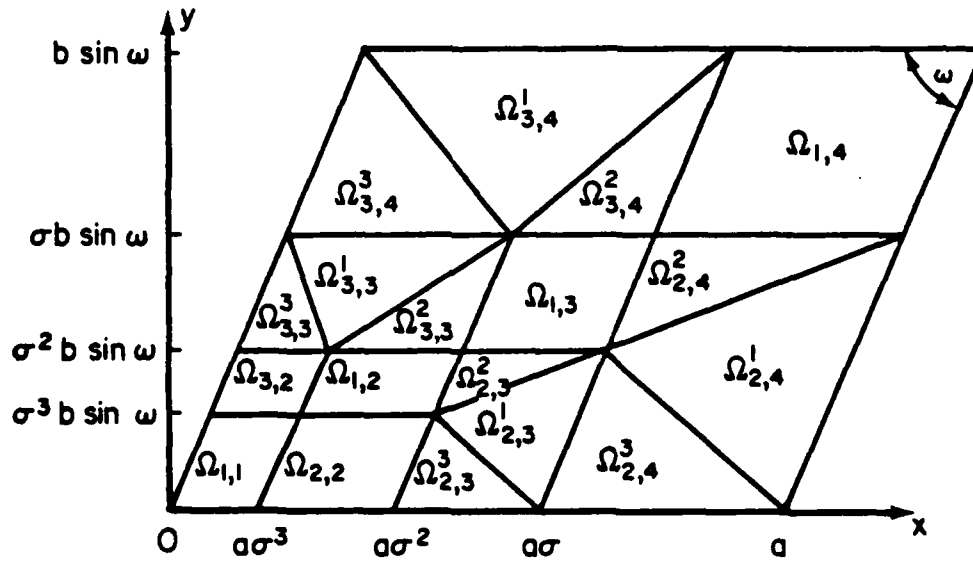


Figure 5.6

Geometric Mesh on a Parallelogram

## 6. GENERALIZED GEOMETRIC MESH AND ITS APPROXIMATION PROPERTIES

In this chapter the generalized geometric mesh with triangular and quadrilateral, curvilinear triangular and curvilinear quadrilateral elements will be employed on polygonal domain  $\Omega$  contained in the unit disk centered at the origin which coincides with one of the vertices of  $\Omega$ . We need to redefine mesh  $\Omega_{\sigma}^n$  and space  $S^{P,Q,1}(\Omega_{\sigma}^n)$ .

Let  $\Omega^n = \{\Omega_{i,j}, j = 1, 2, \dots, n+1, i = 1, 2, \dots, I(j)\}$  be a partition of  $\Omega$  satisfying the following conditions:

1)  $\Omega_{i,j}$ 's are quadrilateral or triangles (curvilinear quadrilaterals or triangles). The intersection of any two  $\Omega_{i,j}$ 's is one common vertex, or one entire common side or is empty;

2) Let  $h_{i,j}$  and  $\underline{h}_{i,j}$  be the maximum length and the minimum length of sides of  $\Omega_{i,j}$  and for all  $i, j$  there is a constant  $\lambda$  such that

$$(6.1) \quad h_{i,j} / \underline{h}_{i,j} \leq \lambda;$$

3) Let  $M = (M_{i,j}, 1 \leq i \leq I(j), 1 \leq j \leq n+1)$  in which  $M_{i,j}$  is a one-to-one mapping of standard square  $S = (0,1) \times (0,1)$  (resp. standard triangle  $T = \{(\xi, \eta) \mid 0 < \eta < 1 - \xi, 0 < \xi < 1\}$ ) onto  $\Omega_{i,j}$ . If  $P_{\ell}$  and  $\gamma_{\ell}$  denote the vertices and sides of  $\Omega_{i,j}$ , then  $M_{i,j}^{-1}(P_{\ell})$  and  $M_{i,j}^{-1}(\gamma_{\ell})$  are the vertices and sides of  $S$ ,  $1 \leq \ell \leq 4$ . Moreover, if  $M_{i,j}$  and  $M_{m,k}$  map standard square onto two elements  $\Omega_{i,j}$  and  $\Omega_{m,k}$  with common side  $\gamma = A_1 A_2$  then for any  $A \in \gamma$ ,  $\text{dist}(M_{i,j}^{-1}(A), M_{i,j}^{-1}(A_{\ell})) = \text{dist}(M_{m,k}^{-1}(A), M_{m,k}^{-1}(A_{\ell}))$ ,  $1 \leq \ell \leq 2$ . It is assumed that the mapping can be written in the form

$$(6.2) \quad \begin{cases} x = X_{i,j}(\xi, \eta) \\ y = Y_{i,j}(\xi, \eta) \end{cases} \quad \text{on } S(\text{resp. } T)$$

with  $X_{i,j}$  and  $Y_{i,j}$  being smooth functions on  $S$ ,

$$(6.3) \quad |D^\alpha x|, |D^\alpha y| \leq Ch_{i,j}, \quad |\alpha| \leq 2$$

and

$$(6.4) \quad C_1 h_{i,j}^2 \leq J_{i,j} \leq C_2 h_{i,j}^2$$

where  $J_{i,j}$  is the Jacobian and the constants  $C$ ,  $C_1$  and  $C_2$  are independent of  $i, j$ .

The mesh  $\Omega_\sigma^n$  ( $0 < \sigma < 1$ ) is called the geometric mesh if in addition the following condition is satisfied:

4) If  $d_{i,j}$  denote the distance between the origin and  $\Omega_{i,j}$ , then

$$(6.5) \quad \sigma^{n+2-j} \leq d_{i,j} < \sigma^{n+1-j}, \quad 1 < j \leq n+1, \quad 1 \leq i \leq I(j),$$

$$(6.6) \quad d_{i,1} = 0, \quad 1 \leq i \leq I(1),$$

$$(6.7) \quad \kappa_1 d_{i,j} \leq \underline{h}_{i,j} \leq h_{i,j} \leq \kappa_2 d_{i,j} \quad \text{for } 1 \leq i \leq I(j), \quad 1 < j \leq n+1,$$

$$(6.8) \quad \kappa_3 \sigma^{n+1} \leq \underline{h}_{i,1} \leq h_{i,1} \leq \kappa_4 \sigma^n \quad \text{for } 1 \leq i \leq I(1)$$

where  $\kappa_i$ ,  $1 \leq i \leq 4$  are the positive constants independent of  $i$  and  $j$ .

Let  $\underline{P} = (p_{i,j}, 1 \leq i \leq I(j), 1 \leq j \leq n+1)$  and  $\underline{Q} = (q_{i,j}, 1 \leq i \leq I(j), 1 \leq j \leq n+1)$  be the degree vectors with integers  $p_{i,j}, q_{i,j} \geq 0$ .

We define the subspace  $S^{\underline{P}, \underline{Q}}(\Omega_\sigma^n)$  as in [9]

$S_{\sigma}^{P,Q}(\Omega^n) = \{\phi \mid \phi(x,y) = \phi_{i,j}(M_{i,j}^{-1}(x,y)) \text{ for } x \in \Omega_{i,j}, \phi_{i,j}(\xi,\eta)$   
 is the polynomial of degree  $\leq p_{i,j}$  in  $\xi$  and of degree  
 $\leq q_{i,j}$  in  $\eta$  on  $S\}$

and  $S_{\sigma}^{P,Q,1}(\Omega_{\sigma}^n) = S_{\sigma}^{P,Q}(\Omega_{\sigma}^n) \cap H^1(\Omega)$ ,  $\dot{S}_{\sigma}^{P,Q,1}(\Omega_{\sigma}^n) = S_{\sigma}^{P,Q,1}(\Omega_{\sigma}^n) \cap \dot{H}^1(\Omega)$   
 where  $\dot{H}^1(\Omega) = \{u \mid u \in H^1(\Omega), u|_{\partial\Omega} = 0\}$ .

Remark 1. Denoting  $U = u(M_{i,j}(\xi,\eta))$ , (6.3) implies that for  $|\alpha| = 1$

$$(6.9) \quad |D^{\alpha}u| \leq Ch_{i,j}^{-1}(|U_{\xi}| + |U_{\eta}|) \text{ on } S,$$

and for  $|\alpha| = 2$

$$(6.10) \quad |D^{\alpha}u| \leq Ch_{i,j}^{-2}(|U_{\xi\xi}| + |U_{\eta\eta}| + |U_{\xi\eta}| + |U_{\eta}| + |U_{\xi}|).$$

Remark 2. If  $\Omega_{i,j}$  is a triangle, (6.3) and (6.4) are equivalent to the well known angle condition:

$$(6.11) \quad 0 < \omega_0 \leq \omega \leq \pi - \omega_0 < \pi \text{ for all interior angles } \omega \text{'s of } \Omega_{i,j}.$$

Remark 3. The geometric mesh  $\Omega_{\sigma}^n$  was designed for the approximation of the functions  $u \in B_{\beta,d}^2(\Omega)$ ,  $\phi_{\beta} = r^{\beta}$ . In an obvious way the mesh can be designed for the approximation of functions  $u \in B_{\beta,d}^2(\Omega)$ ,  $\phi_{\beta} = \prod_{i=1}^M r_i^{\beta_i}$

Lemma 6.1. Let  $\Omega$  be a curvilinear quadrilateral (resp. triangle),  $h$  be the length of the longest arc, and let  $M$  be a one-to-one mapping of standard square  $S = (0,1) \times (0,1)$  (resp. standard triangle  $T = \{(\xi,\eta) \mid 0 \leq \xi \leq \eta, 0 < \xi < 1\}$ ) onto  $\Omega$  given by

$$(6.12) \quad \begin{cases} x = \phi(\xi, \eta) \\ y = \psi(\xi, \eta). \end{cases} \quad \text{on } S \text{ (resp. } T)$$

Assume that for any  $\alpha$ ,  $|\alpha| \geq 1$

$$(6.13) \quad |D^\alpha \phi|, |D^\alpha \psi| \leq A |\alpha| h.$$

Then for  $U(\xi, \eta) = u(\phi(\xi, \eta), \psi(\xi, \eta))$  and  $|\alpha| = k$

$$(6.14) \quad \|D^\alpha U\|_{H^0(S) \text{ (resp. } H^0(T))} \leq (3A)^k \sum_{\ell=1}^k \max_{|\mu|=\ell} \{ \|D^\mu u\|_{H^0(\Omega)} \} \frac{h^\ell (k-1)!}{(\ell-1)!}$$

where  $\mu$  is the multi-index  $(\mu_1, \mu_2)$ .

Proof. Let us first show that for  $|\alpha| = k \geq 1$

$$(6.15) \quad D^\alpha U = \sum_{\ell=1}^k \sum_{\bar{\mu} \in S^\ell} (D^{\bar{\mu}} u) \rho_{\bar{\mu}, \alpha}^{(k)}$$

where  $S^\ell$  is a set of pairs  $\bar{\mu} = (\mu_p, \mu_q)$   $\mu_p, \mu_q \geq 0$  integral,  $\mu_p + \mu_q = k$  such that

$$S^1 = (1, 0) \cup (0, 1)$$

$$(6.16) \quad S^{k+1} = \{(\mu_p, \mu_q) \mid (\mu_p-1, \mu_q) \in S^k, (\mu_p, \mu_q-1) \in S^k\}$$

$S^k$  consists of pairs which are in general repeating. For example

$$S^2 = \{(2, 0), (1, 1), (1, 1), (0, 2)\}$$

$$S^3 = \{(3, 0), (2, 1), (2, 1), (1, 2), (2, 1), (1, 2), (1, 2), (0, 3)\}.$$

The values  $(\mu_p, \mu_q)$  are the numerical values of the pair  $\bar{\mu}$  and the different pairs can have the same numerical value. Further we will



denote  $|\bar{\mu}| = \mu_p + \mu_q$ .

The functions  $\rho_{\bar{\mu}, \alpha}^{(k)}$  depend on the pair  $\bar{\mu}$  (not only its numerical value). For example for  $\ell = 2, \alpha = (1, 1)$  we have in (6.15)

$$\sum_{\bar{\mu} \in S^2} (D^{\bar{\mu}} u) \rho_{\bar{\mu}, \alpha}^{(2)} = u_{xx} \phi_{\xi} \phi_{\xi} + u_{xy} \phi_{\xi} \psi_{\eta} + u_{xy} \psi_{\xi} \phi_{\eta} + u_{yy} \psi_{\xi} \psi_{\eta}$$

and hence the function  $\rho_{\bar{\mu}, \alpha}^{(2)}$  associated to the two different pairs with the same numerical values  $(1, 1)$  are different. In (6.15) the general form of  $\rho_{\bar{\mu}, \alpha}^{(k)}$  is

$$(6.17) \quad \rho_{\bar{\mu}, \alpha}^{(k)} = \sum_{\substack{s_1 + s_2 = |\bar{\mu}| \\ |v_j| + |\kappa_j| = k}} a(\bar{\mu}, v_j, \kappa_j, \alpha) \prod_{j=1}^{s_1} D^{v_j} \phi \prod_{j=1}^{s_2} D^{\kappa_j} \psi$$

with  $a(\mu, v_j, \kappa_j, \alpha) = 0$  or  $1$ . (We used in (6.17) the notation  $\prod_{j=1}^0 = 1$ ).

(6.15) can be readily proven by induction. Let  $\Phi(k, \ell), k \geq \ell$  be the number of (additional) terms in (6.17). It is immediate that

$$(6.18) \quad \Phi(k, k) = 1$$

and

$$(6.19) \quad \Phi(k+1, \ell) = \ell \Phi(k, \ell) + 2\Phi(k, \ell-1)$$

and (6.18) (6.19) gives

$$\Phi(k, \ell) \leq 3^k \ell^{k-\ell}.$$

Hence coming back to (6.17) we get by (6.13)

$$|\rho_{\bar{\mu}, \alpha}^{(k)}| \leq \Phi(k, |\bar{\mu}|) h^{|\bar{\mu}|} A^k$$

and hence

$$\begin{aligned} \|D^\alpha u\|_{H^0(S)} &\leq (3A)^k \sum_{\ell=1}^k \max_{|\mu|=\ell} \{ \|D^\alpha u\|_{H^0(\Omega)} \} h_\ell^{\ell, k-\ell} \\ &\leq (3A)^k \sum_{\ell=1}^k \max_{|\mu|=\ell} \{ \|D^\alpha u\|_{H^0(\Omega)} \} h_\ell^{\ell} \frac{(k-1)!}{(\ell-1)!}. \end{aligned}$$

□

**Theorem 6.1.** Let  $u \in B_{\beta,d}^2(\Omega)$  with  $\phi_\beta = r^\beta$ ,  $0 < \beta < 1$ , and  $\Omega$  be a polygonal domain contained in a unit disc and with a vertex at the origin.

Let  $\Omega_\sigma^n = \{\Omega_{i,j}, 1 \leq i \leq I(j), 1 \leq j \leq n+1\}$  be a curvilinear quadrilateral geometric mesh satisfying conditions 1)-4). Assume that mapping  $M_{i,j}$  of  $S = (0,1) \times (0,1)$  onto  $\Omega_{i,j}$

$$(6.20) \quad \begin{cases} x = \phi_{i,j}(\xi, \eta) \\ y = \psi_{i,j}(\xi, \eta) \end{cases} \quad \text{in } S$$

is such that

(6.21)

$$|D^\alpha \phi_{i,j}|, |D^\alpha \psi_{i,j}| \leq A^k h_{i,j} \quad \text{for any } |\alpha| = k \geq 1, 1 \leq i \leq I(j), 1 \leq j \leq n+1$$

where  $A$  is independent of  $i, j$ , and  $k$ . Then there exists a function

$\phi(x,y) \in S^{\underline{P}, \underline{Q}, 1}(\Omega_\sigma^n)$  with degree vectors  $\underline{P}$  and  $\underline{Q}$  in which  $p_{i,j} = q_{i,j} = p_j = \max(2, [j\mu])$ ,  $j \geq 2$ , and with  $\mu$  satisfying (5.15),  $p_{i,1} = q_{i,1} = 1$  such that

$$(6.22) \quad \|u - \phi\|_{H^1(\Omega)} \leq C e^{-bN^{1/3}}$$

with  $C$  and  $b$  independent of  $N$ .

Proof. For each  $(i,j)$ ,  $U_{i,j}(\xi,\eta) = u(\phi_{i,j}(\xi,\eta), \psi_{i,j}(\xi,\eta))$  is defined on  $S = (0,1) \times (0,1)$  and by (6.21) and Lemma 6.1 the inequality (6.14) holds. Applying Lemma 4.3 there is a polynomial  $\tilde{\phi}_{i,j}(\xi,\eta)$  of degree  $p_j$  in  $\xi$  and  $\eta$  such that for the integer  $t_j$ ,  $1 \leq t_j \leq p_j$  and  $0 \leq m \leq 2$ ,  $j \geq 2$

$$\begin{aligned} \|D^m(U_{i,j} - \tilde{\phi}_{i,j})\|_{H^0(S)} &\leq C \frac{(p_j - t_j)!}{(p_j + t_j + 2 - 2m)!} \left(\frac{1}{2}\right)^{2(t_j+1)} \sum_{k=0}^{t_j+1+k} \left\| \frac{\partial^{t_j+1+k} U_{i,j}}{\partial \xi^{t_j+1} \partial \eta^k} \right\|_{H^0(S)}^2 \left(\frac{1}{2}\right)^{2k} \\ &\quad + \left\| \frac{\partial^{t_j+1+k} U_{i,j}}{\partial \xi^k \partial \eta^{t_j+1}} \right\|_{H^0(S)}^2 \left(\frac{1}{2}\right)^{2k} \end{aligned}$$

by (6.14)

$$\leq C \frac{(p_j - t_j)!}{(p_j + t_j + 2 - 2m)!} \left(\frac{3A}{2}\right)^{2(t_j+1)} \sum_{\ell=1}^{t_j+3} h_{i,j}^{2\ell} \left(\frac{(t_j+2)!}{(\ell-1)!}\right)^2 \|D^\ell u\|_{H^0(\Omega_{i,j})}^2$$

by (6.7) and because  $d_{i,j} < 1$

$$\leq C \frac{(p_j - t_j)! t_j}{(p_j + t_j + 2 - 2m)!} \left(\frac{3A}{2}\right)^{2(t_j+1)} d_{i,j}^{2(2-\beta)} \sum_{\ell=2}^{t_j+3} \left(\frac{(t_j+2)!}{(\ell-1)!}\right)^2 \kappa_2^{2\ell} \|u\|_{H_\beta^{\ell,2}(\Omega)}^2.$$

Letting  $\phi_{i,j}(x,y) = \tilde{\phi}_{i,j}(M_i^{-1}(x,y))$  for  $j \geq 2$ , we have by the scaling argument

$$\begin{aligned} \|D^m(u - \phi_{i,j})\|_{H^0(\Omega_{i,j})}^2 &\leq \frac{C(p_j - t_j)! t_j}{(p_j + t_j + 2 - 2m)!} \left(\frac{3A}{2}\right)^{2t_j} d_{i,j}^{2(2-m-\beta)} \sum_{\ell=2}^{t_j+3} \kappa_2^{2\ell} \left(\frac{(t_j+2)!}{(\ell-1)!}\right)^2 \|u\|_{H_\beta^{\ell,2}(\Omega)}^2. \end{aligned}$$

Let  $\hat{H}_\beta^{k,2}(\Omega)$  ( $k \geq 2$ ) be the weighted Sobolev space with norm

$$\|u\|_{\hat{H}_\beta^{k,2}(\Omega)}^2 = \sum_{\ell=2}^k \kappa_2^{2\ell} \left( \frac{(k-1)!}{(\ell-1)!} \right)^2 \|u\|_{\hat{H}_\beta^{\ell,2}(\Omega)}^2.$$

Since  $u \in B_{\beta,d}^2(\Omega)$  then for  $k \geq 2$

$$\|u\|_{\hat{H}_\beta^{k,2}(\Omega)} \leq C d^{\gamma(k-2)} (k-1)!$$

with  $\gamma = \max(1, \kappa_2 d)$ . Let  $T$  be the operator  $\hat{H}_\beta^{k+3,2}(\Omega) \rightarrow H^m(\Omega_{i,j})$  for  $1 \leq k \leq p_j$ ,  $0 \leq m \leq 2$ ,  $Tu = u - \phi_{i,j}$  then

$$M_k = \|T\|_{\hat{H}_\beta^{k+3,2}(\Omega), H^m(\Omega_{i,j})} \leq C \left( \frac{(p_j - k)!}{(p_j + k + 2 - 2m)!} \left( \frac{3A}{2} \right)^{2k} d_{i,j}^{2(2-m-\beta)} \right)^{1/2}.$$

Let  $\hat{H}_\beta^{t_j+3,2}(\Omega) = (\hat{H}_\beta^{k+2,2}(\Omega), \hat{H}_\beta^{k+3,2}(\Omega))_{\theta, \infty}$  be the interpolation space by the K-method [10] for  $t_j = k - 1 + \theta$ . Then  $T$  is linear operator:

$\hat{H}_\beta^{t_j+3,2}(\Omega) \rightarrow H^m(\Omega_{i,j})$  with norm

$$\begin{aligned} M_{t_j}^2 &= \|T\|_{\hat{H}_\beta^{t_j+3,2}(\Omega), H^m(\Omega_{i,j})}^2 \leq \left( M_k^\theta M_{k-1}^{1-\theta} \right)^2 \\ &\leq \frac{\Gamma(p_j - t_j + 1)}{\Gamma(p_j + t_j + 3 - 2m)} \left( \frac{3A}{2} \right)^{2t_j} d_{i,j}^{2(2-m-\beta)}, \end{aligned}$$

(see the proof of Theorem 4.1). Thus for a real number  $t_j$ ,  $1 \leq t_j \leq p_j$

$$(6.23) \quad \|D^m(u - \phi_{i,j})\|_{H^0(\Omega_{i,j})}^2 \leq \frac{\Gamma(p_j - t_j + 1)}{\Gamma(p_j + t_j + 3 - 2m)} \left( \frac{3A}{2} \right)^{2t_j} d_{i,j}^{2(2-m-\beta)} \|u\|_{\hat{H}_\beta^{t_j+3,2}(\Omega)}^2.$$

By (2.3)

$$(6.24) \quad \|u\|_{\hat{H}_\beta^{t_j+3,2}(\Omega)} \leq C t_j^{1/2} d_{i,j}^{t_j+1} \Gamma(t_j+2).$$

For  $j = 1$ ,  $\tilde{\phi}_{i,1}$  is always chosen as bilinear interpolation of  $U_{i,1}$  at vertices of  $\Omega_{i,1}$ . Letting  $\phi_{i,1} = \tilde{\phi}_{i,1}(M_{i,1}^{-1}(x,y))$  we have by Lemma 3.6 and the standard scaling argument

$$\begin{aligned} \|u - \phi_{i,1}\|_{H^1(\Omega_{i,1})} &\leq C \|U_{i,1} - \tilde{\phi}_i\|_{H^1(S)} \\ &\leq C |U_{i,1} - \tilde{\phi}_i|_{H_\beta^{2,2}(S)} \end{aligned}$$

where  $\tilde{\phi}_\beta = \tilde{r}^\beta$ ,  $\tilde{r}$  is the distance between point  $(\xi, \eta) \in S$  and  $M_{i,1}^{-1}(0,0)$ , and

$$(6.25) \quad \|u - \phi_{i,1}\|_{H^1(\Omega_{i,1})} \leq Ch_{i,1}^{1-\beta} \|u\|_{H_\beta^{2,2}(\Omega_{i,1})}.$$

To achieve continuity of the polynomials on two adjacent elements we use the same procedure as we used in the previous chapter. We construct in this way the function  $\phi \in S^{\underline{P}, \underline{Q}, 1}(\Omega_0^n)$  such that

$$\begin{aligned} (6.26) \quad \|u - \phi\|_{H^1(\Omega)}^2 &\leq C \left\{ \sum_{i=1}^{I(1)} h_{i,1}^{2(1-\beta)} |u|_{H_\beta^{2,2}(\Omega)}^2 \right. \\ &\quad + \sum_{\substack{1 \leq j \leq I(j) \\ 1 < j \leq n+1}} \left( \frac{3A}{2} \right)^{2t_j} \frac{\Gamma(p_j - t_j + 1)}{\Gamma(p_j + t_j - 1)} d_{i,j}^{2(1-\beta)} \|u\|_{\hat{H}_\beta^{t_j+3,2}(\Omega)}^2 \Big\} \\ &\leq C \sigma^{2(1-\beta)(n+2)} \left\{ I(1) + \sum_{j=2}^{n+1} I(j) \frac{|F(3Ad, \alpha)|^{p_j}}{\sigma^{2(1-\beta)j}} p_j^6 \right\} \end{aligned}$$

where  $t_j = \alpha_j p_j$ , and  $F(d_1, \alpha) = \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}} \left( \frac{\alpha d_1}{2} \right)^{2\alpha}$ . It has been proved that

$$\min_{\alpha} F(d_1, \alpha) = F(d_1, \frac{2}{\sqrt{4+d_1^2}}) = F_{\min} < 1.$$

It can also be shown by (6.5), (6.7) and (6.8) that  $I(j) \leq K$ , for  $1 \leq j \leq n+1$  and some  $K > 0$  independent of  $N$ . Letting

$$\alpha_j = \max\{\frac{1}{p_j}, \frac{2}{\sqrt{4+(3Ad)^2}}\}, \quad p_j = \max(2, [j\mu]), \quad j \geq 2 \text{ with } \mu \text{ satisfying}$$

(5.15) we have from (6.26)

$$\begin{aligned} \|u-\phi\|_{H^1(\Omega)}^2 &\leq CK\sigma^{2(1-\beta)(n+2)} \\ &\leq \tilde{C}e^{-2(1-\beta)(\frac{3}{K\bar{\mu}^2})^{1/3} \ln(\frac{1}{\sigma}) \cdot N^{1/3}} \end{aligned}$$

with  $\tilde{C}$  independent of  $N$  and  $\bar{\mu} = \max(1, \mu)$ . □

**Theorem 6.2.** Let  $u \in B_{\beta,d}^2(\Omega)$  with  $\phi_{\beta} = r^{\beta}$ ,  $0 < \beta < 1$ , and  $\Omega$  be a polygonal domain  $\Omega$  contained in a unit disc and with a vertex at origin. Let  $\Omega_{\sigma}^n = \{\Omega_{i,j}, 1 \leq i \leq I(j), 1 \leq j \leq n+1\}$  by a curvilinear triangular geoemtric mesh satisfying conditions 1)-4) in Section 4.4. Assume that the mapping  $M_{i,j}$  of  $T = \{(\xi, \eta) \mid 0 < \eta, 0 < \xi, \xi + \eta < 1\}$  onto  $\Omega_{i,j}$

$$\begin{cases} x = \phi_{i,j}(\xi, \eta) \\ y = \psi_{i,j}(\xi, \eta) \end{cases} \quad \text{in } T$$

is such that

$$|D^{\alpha} \phi_{i,j}|, |D^{\alpha} \psi_{i,j}| \leq A |\alpha|_{h_{i,j}} \quad \text{for any } |\alpha| = k \geq 1, \quad 1 \leq i \leq I(j), \quad 1 \leq j \leq n+1,$$

where  $A$  is independent of  $i, j$  and  $k$ . In addition assume that  $M_{i,j}$  can be extended to standard square  $S = (0,1) \times (0,1)$ ,  $M_{i,j}(S) \subset \Omega$  preserving the properties of  $M_{i,j}$  on  $T$  and that

$$\tilde{d}_{i,j} = \text{dist}(M_{i,j}(S), 0) \geq \kappa_0 d_{i,j} \quad \text{for all } i \text{ and } j$$

with constant  $\kappa_0 > 0$ . Then there exists  $\phi(x, y) \in (S^{\underline{P}, \underline{Q}}(\Omega_\sigma^n) \oplus S^{\tilde{\underline{P}}, \tilde{\underline{Q}}}(\Omega_\sigma^n)) \cap H^1(\Omega)$  with degree vectors  $\underline{P}$  and  $\underline{Q}$  in which  $p_{i,j} = q_{i,j} = p_j = p_j = \max(2, [j\mu])$  for  $j \geq 2$  and  $\mu$  satisfying (5.15),  $p_{i,1} = q_{i,1} = 1$ , and with degree vectors  $\tilde{\underline{P}}$  and  $\tilde{\underline{Q}}$  in which  $\tilde{p}_{i,j} = 1$ ,  $\tilde{q}_{i,j} = 2p_j - 1$  such that

$$\|u - \phi\|_{H^1(\Omega)} \leq C e^{-bN^{1/3}}$$

with  $C$  and  $b$  independent of  $N$ .

Proof. The proof is analogous to the proof of Theorem 6.1.

Remark 1. We assumed in Theorem 6.2 that the mapping can be extended. In contrast to the mapping of  $T$  where one-to-one mapping is assumed together with condition 4) we do not need these assumptions to be satisfied on the entire  $S$ , we need only that  $M_{i,j}(S) \subset \Omega$ .

Remark 2. Assume that  $\Omega$  is a simple parallelogram or triangle only and assume that  $u \in B_{0,d}^2$  but such that the domain of analyticity can be extended into a small neighborhood of  $\Omega$ . If  $\Omega$  is a parallelogram and the mesh consists of one element only then the rate of the  $p$ -version is exponential. It follows immediately from our analysis. If  $\Omega$  is a triangle and the domain consists of one (triangle) element only then we cannot conclude directly from our theory that the rate of convergence is exponential.

Remark 3. In Theorems 6.1 and 6.2 we assumed that the domain  $\Omega$  is a polygon. Obviously this assumption is not essential.

Remark 4. As in the previous chapters we can consider in Theorems 6.1 and 6.2 uniform degree of elements and get the exponential rate of convergence too.

Remark 5. The quadrilateral is the special case of a curvilinear quadrilateral. Suppose that  $(x_1, y_1)$  are the vertices of  $\Omega$ . The mapping of  $S = (0,1) \times (0,1)$  onto  $\Omega$  is

$$\begin{cases} x = \phi(\xi, \eta) = x_1 + (x_2 - x_1)\xi + (x_4 - x_1)\eta + (x_1 - x_2 + x_3 - x_4)\xi\eta \\ y = \psi(\xi, \eta) = y_1 + (y_2 - y_1)\xi + (y_4 - y_1)\eta + (y_1 - y_2 + y_3 - y_4)\xi\eta. \end{cases}$$

Obviously (6.21) holds.

Remark 6. In the mesh shown in Fig. 6.1 the curvilinear triangular and curvilinear quadrilateral elements are combined. This kind of mesh is important in practice.

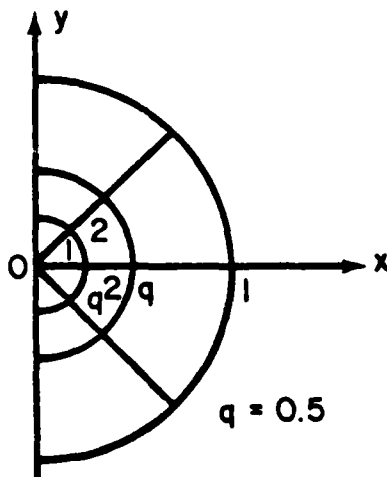


Figure 6.1

Curved Geometric Mesh

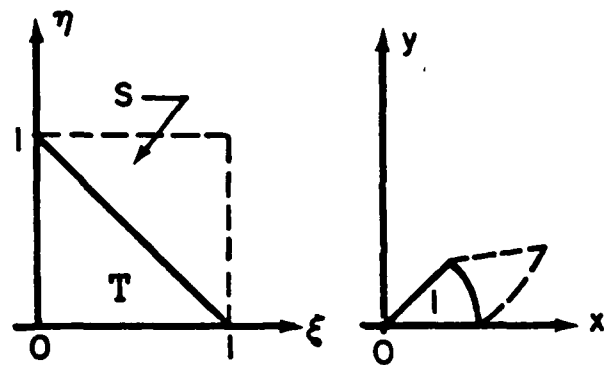


Figure 6.2

Mapping of Standard Triangle



The mapping of  $S = (0,1) \times (0,1)$  onto the curvilinear quadrilateral element  $M_1$  is

$$\begin{cases} x = \phi(\xi, \eta) = \frac{4}{\pi} h(\xi(1-q)+q) \cos \frac{\eta\pi}{4} \\ y = \psi(\xi, \eta) = \frac{4}{\pi} h(\xi(1-q)+q) \sin \frac{\eta\pi}{4} \end{cases}$$

with  $h = q\pi/4$  and (6.21) is satisfied.

The mapping of  $T$  onto curvilinear triangle (element 2) can be defined in various ways. Let us mention one

$$\begin{cases} x = \phi(\xi, \eta) = q^2(\eta \cos \frac{\pi}{4} + \frac{\xi}{1-\eta} (\cos \frac{\pi\eta}{4} - \eta \cos \frac{\pi}{4})) \\ y = \psi(\xi, \eta) = q^2(\eta \sin \frac{\pi}{4} + \frac{\xi}{1-\eta} (\sin \frac{\pi\eta}{4} - \eta \sin \frac{\pi}{4})). \end{cases}$$

Observing that

$$(\cos \frac{\pi\eta}{4} - \eta \cos \frac{\pi}{4}) \frac{1}{1-\eta}$$

and

$$(\sin \frac{\pi\eta}{4} - \eta \sin \frac{\pi}{4}) \frac{1}{1-\eta}$$

is analytic at  $\eta = 1$  and we easily see that with  $q^2 = h$  (6.21) is satisfied. It is also easy to see that  $M(S) \subset \Omega$  (see Fig. 6.2).

Remark 7. Consider an element  $\Omega$  with vertices  $(x_i, y_i)$ ,  $1 \leq i \leq 4$  and curvilinear edges  $\gamma_i$  (see Fig. 6.3),  $\gamma_i$ 's are described by

$$\begin{cases} \tilde{x}_i = g_i(\xi) \\ \tilde{y}_i = f_i(\xi) \end{cases} \quad \text{for } 0 \leq \xi \leq 1, i = 1, 3$$

and

$$\begin{cases} \tilde{x}_i = g_i(\eta) \\ \tilde{y}_i = f_i(\eta) \end{cases} \quad \text{for } 0 \leq \eta \leq 1, i = 2, 4$$

where  $g_i$  and  $f_i$ ,  $1 \leq i \leq 4$  are infinitely differentiable, and

$$|g_i^{(k)}|, |f_i^{(k)}| \leq h A^k, \quad \text{for } 1 \leq i \leq 4, \text{ any } k \geq 0$$

and

$$|g_i - g_{i+2}|, |f_i - f_{i+2}| \leq Ah \quad \text{for } i = 1, 2$$

with  $A \geq 1$  independent of  $k$ ,  $h$  and  $h$  being the arc length of the longest curvilinear edges of  $\Omega$ . Then the mapping of  $S = (0,1) \times (0,1)$  onto  $\Omega$  by a blending function is constructed as follows:

$$\begin{cases} x = \phi(\xi, \eta) = \tilde{x}_1(\xi)(1-\eta) + \tilde{x}_2(\eta)\xi + \tilde{x}_3(\xi)\eta + \tilde{x}_4(\eta)(1-\xi) \\ \quad -x_1(1-\xi)(1-\eta) - x_2\xi(1-\eta) - x_3\xi\eta - x_4\eta(1-\xi) \\ y = \psi(\xi, \eta) = \tilde{y}_1(\xi)(1-\eta) + \tilde{y}_2(\eta) + \tilde{y}_3(\xi)\eta + \tilde{y}_4(\eta)(1-\xi) \\ \quad -y_1(1-\xi)(1-\eta) - y_2\xi(1-\eta) - y_3\xi\eta - y_4\eta(1-\xi). \end{cases}$$

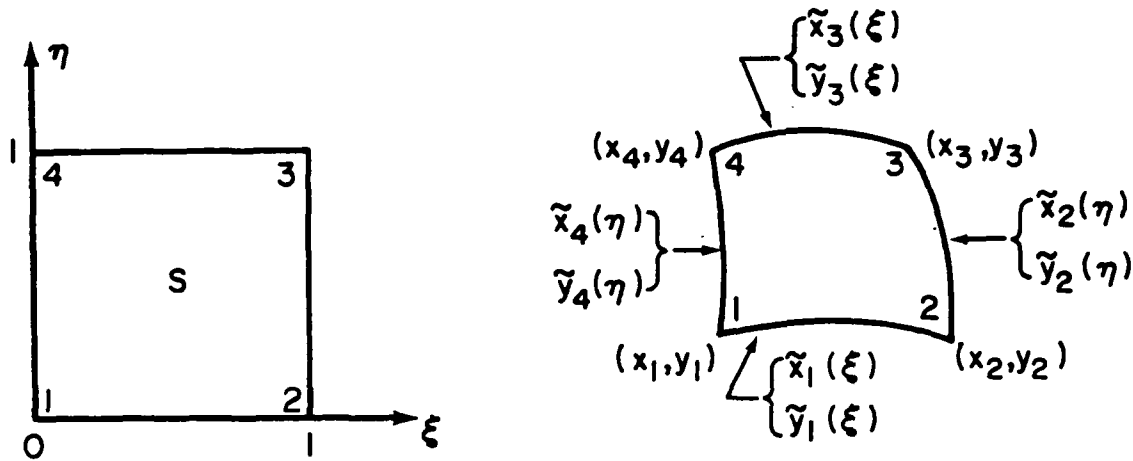


Figure 6.3

Mapping of Standard Square

Then it is easy to see that

$$\phi_{\xi} = (1-\eta)\tilde{x}'_1(\xi) + \eta\tilde{x}'_3(\xi) + \tilde{x}_2(\eta) - \tilde{x}_4(\eta) + (x_1-x_2)(1-\eta) + (x_4-x_3)\eta,$$

$$|\phi_{\xi}| \leq 5Ah;$$

$$\phi_{\xi}^{\ell} = (1-\eta)\tilde{x}_1^{(\ell)}(\xi) + \eta\tilde{x}_3^{(\ell)}(\xi), \quad \phi_{\xi\eta}^{\ell} = -x_1^{(\ell)}(\xi) + x_3^{(\ell)}(\xi)$$

$$|\phi_{\xi}^{\ell}|, |\phi_{\xi\eta}^{\ell}| \leq 2A^{\ell}h, \quad \text{for } \ell \geq 2,$$

$$\phi_{\xi\eta} = -\tilde{x}'_2(\xi) + \tilde{x}'_3(\xi) + (x_2-x_1) + (x_4-x_1), \quad |\phi_{\xi\eta}| \leq 4Ah;$$

$$\phi_{\xi\eta}^{\ell m} = 0 \quad \text{for } m \geq 1, \ell \geq 1 \quad \text{or} \quad \ell \geq 2, m \geq 1.$$

Similarly we can see that

$$|\phi_{\eta^m}|, |\phi_{\xi\eta^m}| \leq 2A^m h \quad \text{for } m \geq 2,$$

$$|\phi_{\eta}| \leq 5Ah.$$

These inequalities are also true for  $\psi$ . Hence (6.21) holds.

Remark 8. In the theorems above we assumed that the singularities are located in the vertices of the domain  $\Omega$  i.e.,  $\phi_{\beta}(x) = \prod_{i=1}^M |x - A_i|^{\beta_i}$  when  $A_i$  are the vertices of  $\Omega$ . Assume now that  $A_i$  are located outside of  $\Omega$ . Then it is possible to show that  $\|e\| \leq Ce^{-a} N^{1/2}$ . This rate of convergence is achieved by the  $p$ -version when the size of the minimal element of the optimal mesh is not going to zero as  $p \rightarrow \infty$ .

## 7. NUMERICAL RESULTS

In this chapter we will discuss the numerical results for the solution of a model plane elasticity problem. We selected the model of a cracked panel loaded by such tractions that the exact solution is the first and second mode of the stress intensity factor solution. This problem was selected because it characterizes the difficulties of the usual engineering problems. We will compare the performance of the  $h$ ,  $p$  and  $h$ - $p$  versions of the finite element method by focussing on the accuracy measured in the energy norm.

The purpose of the numerical computation is the following:

1) Our estimates are upper estimates which have asymptotic character. It is important to see the numerical behavior of the error, its asymptotic range, the size of the constants characterizing the error and the maximal accuracy which is practically achievable.

2) The  $h$ - $p$  version is characterized by the mesh factor  $\sigma$  of the geometric mesh and the degree factor  $\mu$  governing mesh size and the growth of the element degrees, respectively. Numerical results will show the sensitivity of the accuracy on  $\sigma$  and  $\mu$ , the values of the optimal factors  $\sigma$  and  $\mu$  leading to the highest accuracy.

3) Our theory does not allow us to distinguish between performance of elements of various type, the curvilinear and rectangular (resp. triangular). The computer time is smaller for rectangular or triangular elements than for curvilinear elements. Therefore the question arises whether curvilinear elements in the neighborhood of the crack can improve the accuracy because the singularity has radial character.

4) It is known that the  $p$ -version is very insensitive to the size of the Poisson ratio (nearly incompressibility) (see [8], [22]). Numerical solution will show the effect of the Poisson ratio on the accuracy of the

solution in the case of the h-p version.

5) The standard finite element programs are based on the h-version with low order elements. Several hundred finite element programs have already been developed. One of the most popular packages is NASTRAN in its various versions, for example MSC/NASTRAN. Other widely used packages are for example, ADINA, ANSYS, STRUDL, GIFTS, PAFEC, etc.

There is only one code based on the p and the h-p versions, the code PROBE. The architecture of this program is different from the above mentioned. It utilizes the hierarchic type of elements, computes simultaneously solutions of different degree elements, etc. We will make some (crude) comparison of computation by PROBE with the h-version.

The theory and computation addresses only the performance with respect to the error measured in the energy norm. Although in practice other measures are essential, the energy norm performance is obviously the starting point of main theoretical importance.

The computation of the h-p version has been done by the program PROBE of NOETIC TECHNOLOGIES, Inc., St. Louis. (See [21]).

The h-version computation has been done by the adaptive program FEARS developed at the University of Maryland. (See [14] [19]).

We shall consider the plane strain problem of two dimensional elasticity (homogeneous, isotropic material) with  $E$  and  $\nu$  denoting the Young's modulus of elasticity and Poisson ratio respectively ( $E > 0$ ,  $0 \leq \nu < .5$ ). The domain  $D$  under consideration is a square panel with a crack as shown in Fig. 7.1.

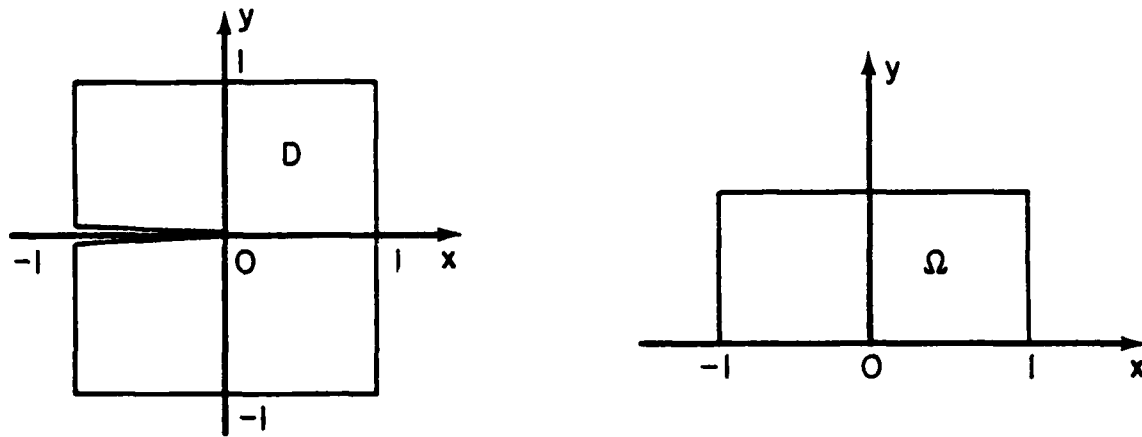


Figure 7.1  
Cracked Panel

Let  $U = (u, v)$  be the displacement vector,  $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$  be the stress tensor and  $T$  be the traction vector. The displacement vector  $U$  satisfies the Lamé-Navier equations

$$(7.1) \quad \begin{cases} -(\lambda + \mu) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \mu \Delta u = 0 \\ -(\lambda + \mu) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \mu \Delta v = 0 \end{cases} \quad \text{in } D$$

and the boundary condition

$$(7.2) \quad T|_{\partial D} = \tau \cdot \vec{n}|_{\partial D} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where  $\vec{n}$  is the unit outside normal to the boundary  $\partial D$ .  $\lambda = \frac{Ev}{(1+\nu)(1-2\nu)}$  and  $\mu = \frac{E}{2(1+\nu)}$  are the Lamé coefficients. The bilinear form associated with (7.1) and (7.2) is

$$(7.3) \quad B(U, W) = \int_D \left\{ \mu \left( \frac{\partial u}{\partial x} \frac{\partial w_1}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial w_2}{\partial y} \right) + \frac{\lambda}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} \right) \right. \\ \left. + \frac{\mu}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left( \frac{\partial w_2}{\partial x} + \frac{\partial w_1}{\partial y} \right) \right\} dx dy$$

with  $U = (u, v)$ ,  $W = (w_1, w_2) \in H^1(D) \times H^1(D)$  and the linear functional

$$F(W) = \int_{\partial\Omega} (f_1 w_1 + f_2 w_2) ds.$$

The weak solution  $U \in H^1(D) \times H^1(D)$  satisfies

$$(7.4) \quad B(U, W) = F(W) \quad \text{for any } W \in H^1(D) \times H^1(D).$$

The strain energy functional  $G(U)$  is

$$(7.5) \quad G(U) = \frac{E}{2(1-2\nu)(1+\nu)} \int_D \left\{ (1-\nu) \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) + 2\nu \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right. \\ \left. + \frac{1-2\nu}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} dx dy \\ = \frac{1}{2} B(U, U).$$

We will consider problem 1 and problem 2 when the imposed tractions lead to the symmetric and antisymmetric mode of the stress intensity solution. For both modes  $i = 1, 2$  the solution has singular behavior at the tip of the crack (see [19]).

$$u_i \approx r^{1/2} \phi_{1,i}(\theta)$$

$$v_i \approx r^{1/2} \psi_{1,i}(\theta)$$

and hence  $u_i, v_i \in H^{3/2-\varepsilon}(\Omega)$ ,  $\varepsilon > 0$ , and  $u_i, v_i \in H_{\beta}^{2,2}(\Omega)$ ,  $\frac{1}{2} < \beta < 1$ .



The solution of Problem 1 has to be augmented by the conditions

$$u_1(0,0) = v_1(0,0) = 0$$

fixing the rigid body motion. For the solution of Problem 2 we impose the conditions

$$u_2(0,0) = v_2(0,0) = v_2(-1,0) = 0$$

fixing the rigid body motion. The strain energy of true solutions  $U_i$  for  $E = 1.0$  and  $\nu = 0.3$  is

$$G(U_1) = 0.6017796916,$$

$$G(U_2) = 0.2370646876.$$

By  $U_0^i$ ,  $i = 1, 2$  we denote the exact solutions and by  $U_{FE}^i$ ,  $i = 1, 2$ , the finite element of Problem 1 and Problem 2. The error of the finite element solution will be

$$(7.6) \quad e_i = U_0^i - U_{FE}^i.$$

The energy norm of the error  $\|e_i\|_E$  is directly related to the strain energy of the exact and finite element solution

$$(7.7) \quad \|e_i\|_E^2 = G(U_0^i) - G(U_{FE}^i) = \frac{1}{2}(B(U_0^i, U_0^i) - B(U_{FE}^i, U_{FE}^i))$$

and the relative error in energy norm is defined as

$$(7.8) \quad \|e_i\|_{E,R} = \frac{\|e_i\|_E}{G(U_0^i)^{1/2}} \cdot 100\%.$$

We used geometric meshes with the factor  $\sigma$  for the  $p$  and  $h$ - $p$  version and studied the performance of two types of meshes A and C shown in Fig. 7.2 and Fig. 7.3.

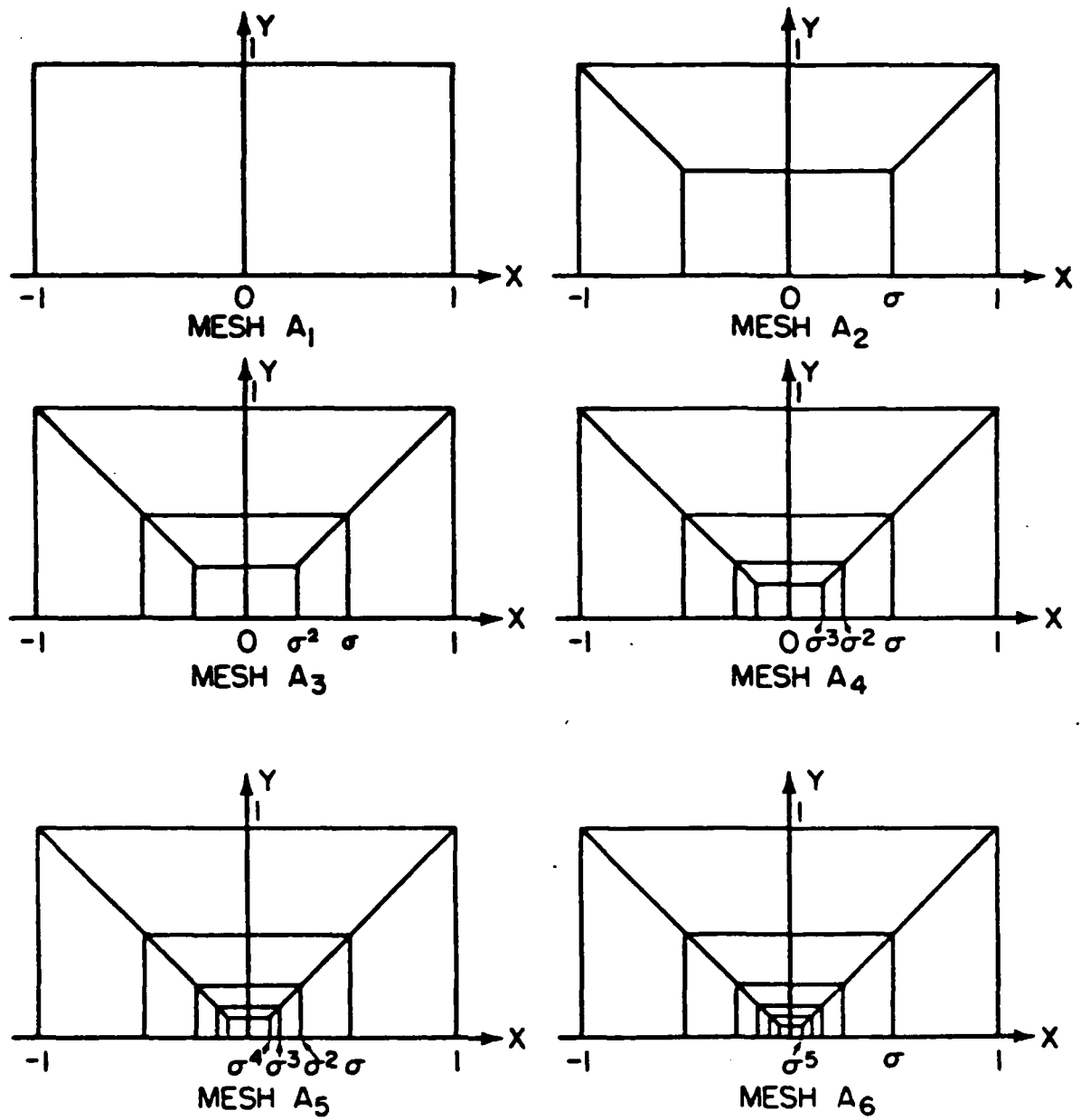


Figure 7.2

Mesh  $A$

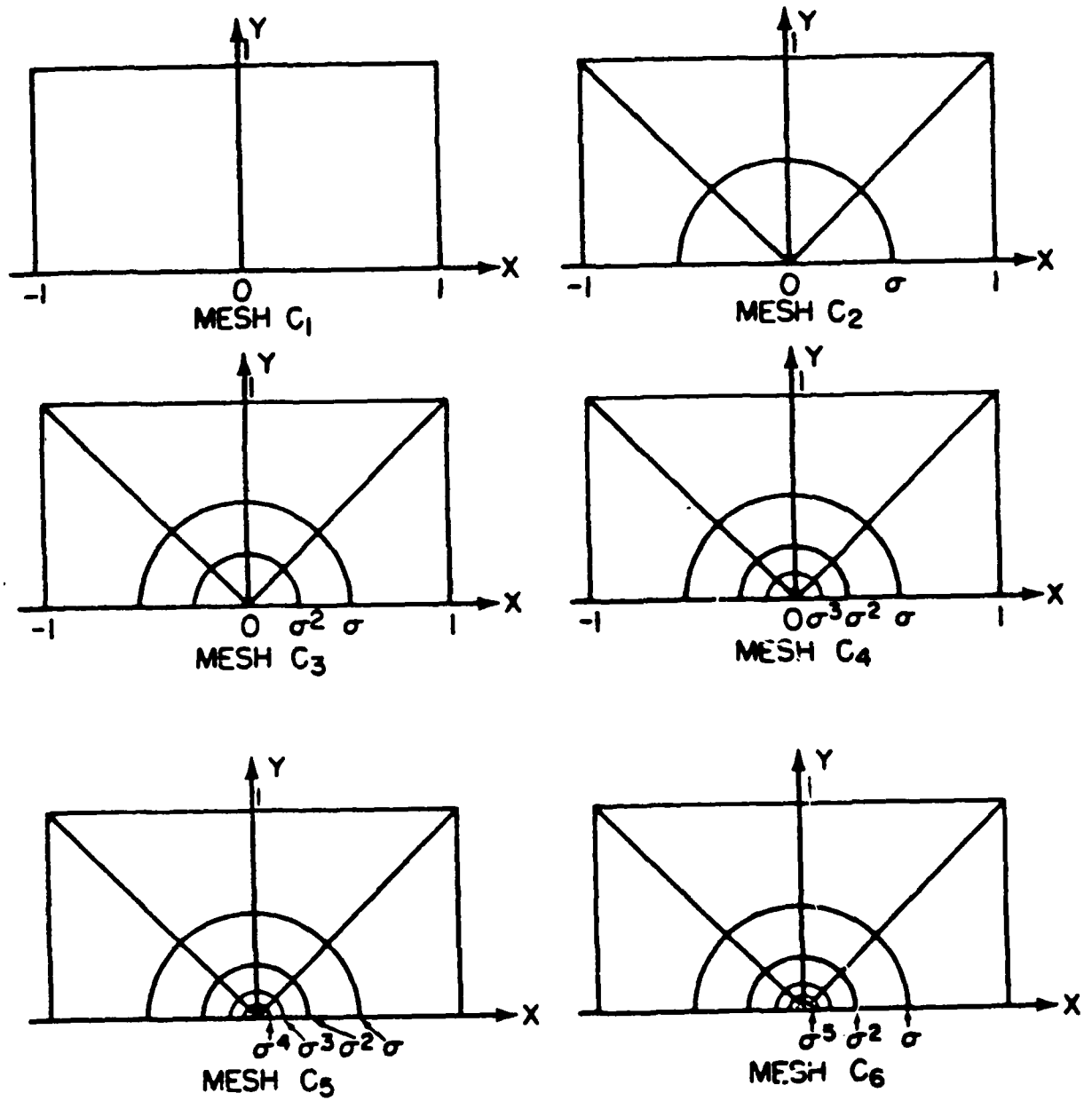


Figure 7.3

Mesh C

Let us mention that the size  $h_{\min}$  of the smallest element is  $\sigma^5$ . For  $\sigma = .15$  we get  $h_{\min} = .759 \times 10^{-4}$  and for  $\sigma = .08$  we get  $h_{\min} = .327 \times 10^{-5}$  with  $h_{\max} = 1$ . The ratio between the size of the largest and smallest element is hence more than  $10^5$ . The computation has been performed by the PROBE program with uniform degree of elements in double precision on Appolo 420 (work length of double precision is 15 decimals). The computation for the h-version was implemented by the program FEARS on UNIVAC 1100 (in single precision). By  $N$  we denote the number of degrees of freedom.

As indicated in the previous chapter there are some constant  $b$  and  $C$  independent of  $N$  such that for the h-p versions we have

$$(7.9) \quad \|e_i\|_{E,R} \leq \frac{C_i}{\|u_0^i\|_E} e^{-b_i N^{1/3}}, \quad i = 1, 2.$$

Table 7.1 and Table 7.2 show the relationship between  $\|e_i\|_{E,R}$ , and  $N$ ,  $p$ ,  $n$ , on mesh A with  $\sigma = .15$  and  $\mu = 1$  (i.e.,  $p = n$ ). The relationships are plotted in Fig. 7.4 and 7.5 on ln-cubic root scale. The curve of the h-p version is almost a straight line which is the envelope of six curves of the p-version for  $1 \leq n \leq 6$ . This means the asymptotic property is achieved already for  $n = p = 2$ . The constants  $b_i$  and  $C_i$ ,  $i = 1, 2$  are numerically given in the tables. For  $p = 6$  we have  $b_1 = 0.670$ ,  $b_2 = 0.668$ , and  $\frac{C_1}{\|u_0^1\|_E} = 1.688$ ,  $\frac{C_2}{\|u_0^2\|_E} = 1.306$ .

In Fig. 7.6 we show the dependence of the error on  $\sigma$ . We see that the best value of  $\sigma$  is close to  $(\sqrt{2}-1)^2 = .17$  which is the theoretically optimal value in one dimension (see [16]).

Fig. 7.6 shows the dependence of the error on  $\mu$  characterizing the relation between  $n$  and  $p$ .

TABLE 7.1

Relationship between  $\|e_1\|_{E,R}^{N,n,p,b_1}$  and  $C_1$  for the h-p Version  
 Problem 1 ( $E=1, \nu=0.3$ ) on Mesh A,  $\sigma=0.15, \mu=1$

Mesh	P	N	$N^{1/3}$	$\ e_1\ _{E,R}$	$b_1$	$C_1 / \ U_0^1\ _E$
$A_1$	1	9	2.08	60.92	.741	1.455
$A_2$	2	48	3.63	20.23	.740	2.303
$A_3$	3	121	4.95	7.61	.776	2.098
$A_4$	4	256	6.35	2.57	.720	1.810
$A_5$	5	477	7.81	.90	.670	1.683
$A_6$	6	808	9.31	.33	.670	1.688

TABLE 7.2

Relationship between  $\|e_2\|_{E,R}^{N,n,p,b_2}$  and  $C_2$  for the h-p Version  
 Problem 2 ( $E=1, \nu=0.3$ ) on Mesh A,  $\sigma=0.15, \mu=1$

Mesh	P	N	$N^{1/3}$	$\ e_2\ _{E,R}$	$b_2$	$C_2 / \ U_0^2\ _E$
$A_1$	1	8	2	43.74	.626	1.664
$A_2$	2	47	3.61	15.97	.742	1.781
$A_3$	3	120	4.93	5.91	.772	1.592
$A_4$	4	255	6.34	2.02	.718	1.395
$A_5$	5	476	7.81	.71	.668	1.309
$A_6$	6	807	9.31	.26	.668	1.306

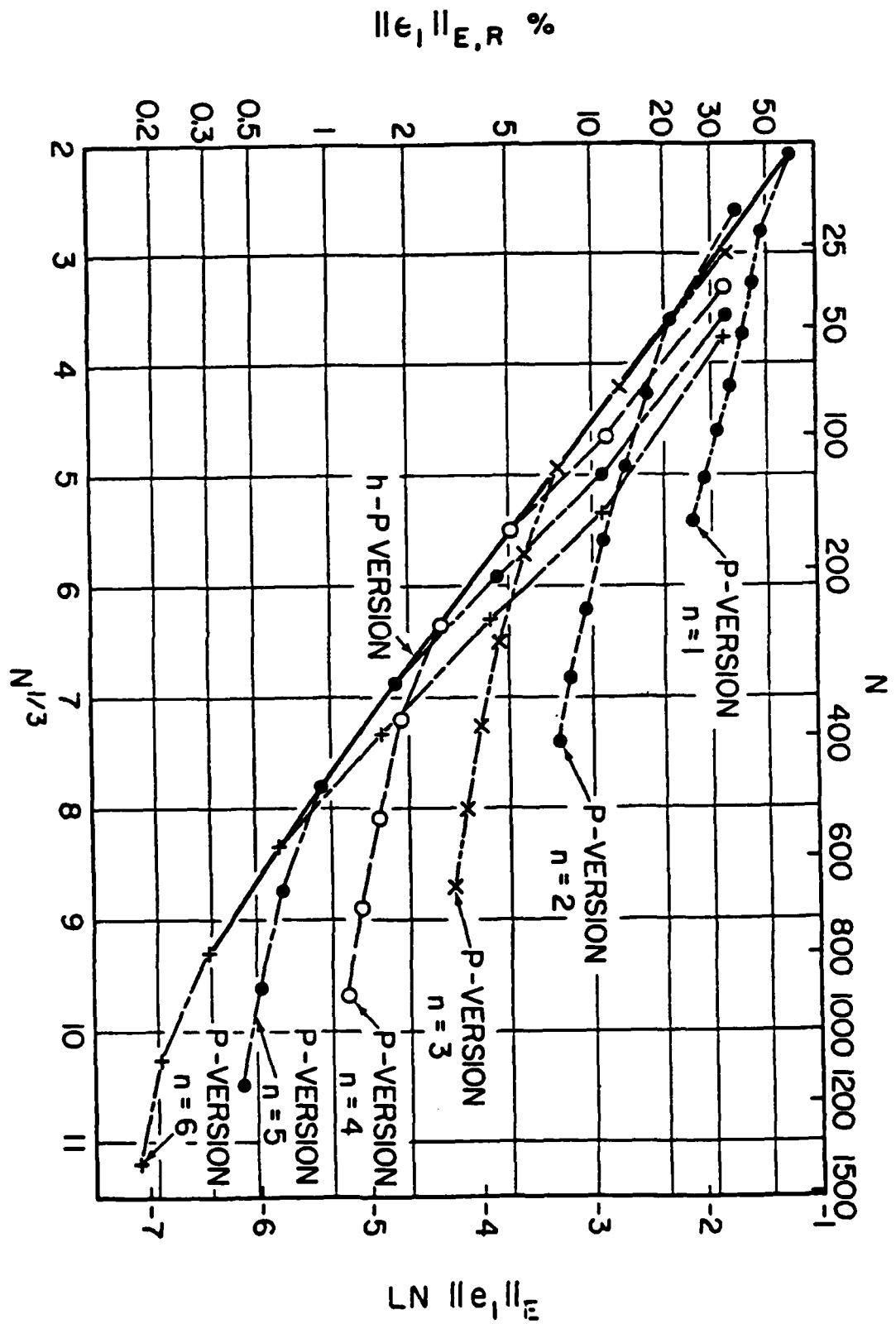


Figure 7.4

Relative Error in Energy Norm vs. Degrees of Freedom  
 Problem 1 ( $E=1, v=.3$ ) on Mesh  $A_n, 1 \leq n \leq 6, \alpha=.15$

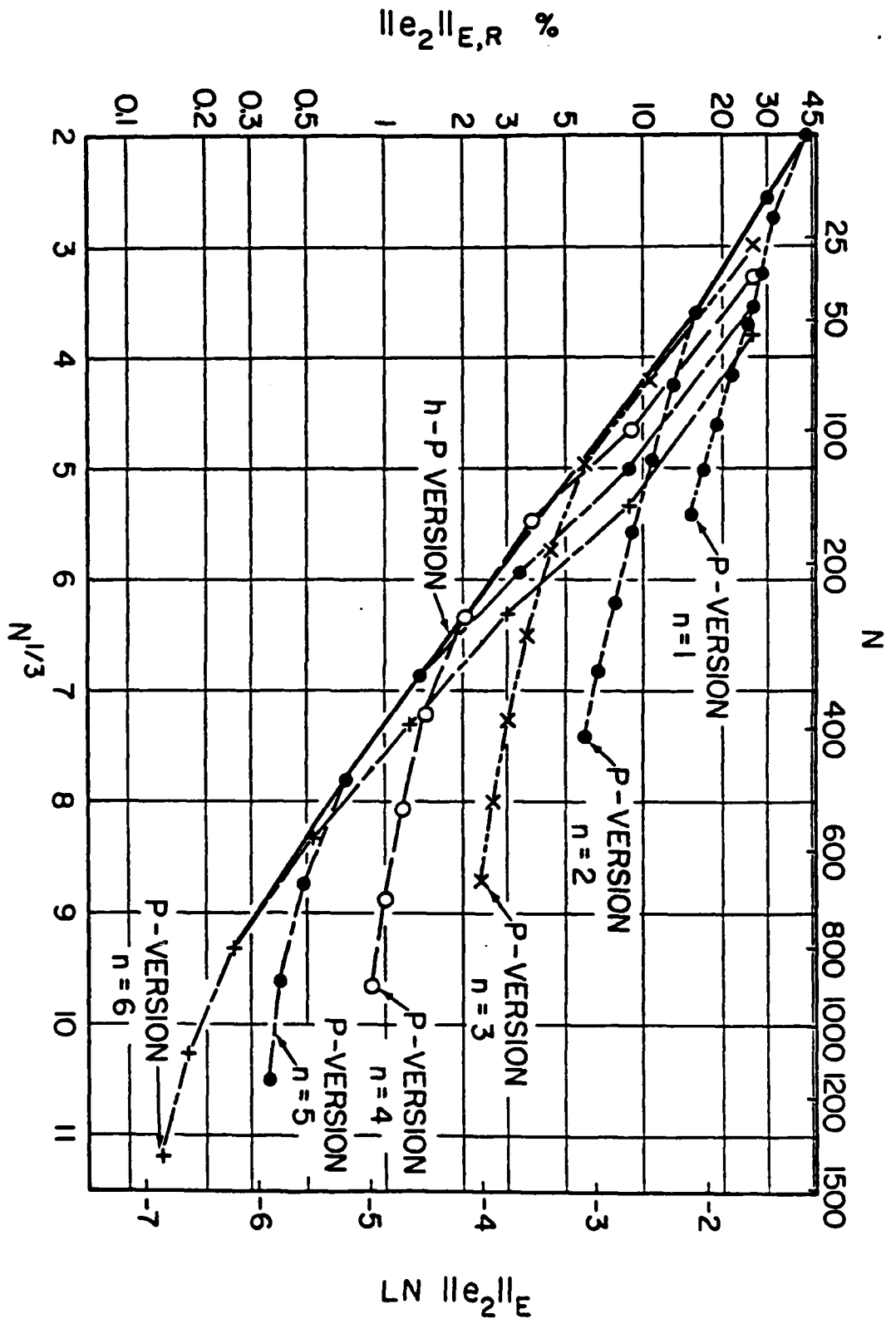


Figure 7.5

Relative Error in Energy Norm vs. Degrees of Freedom  
 Problem 2 ( $E=1, \nu=.3$ ) on Mesh  $A_n, 1 \leq n \leq 6, \alpha=.15$

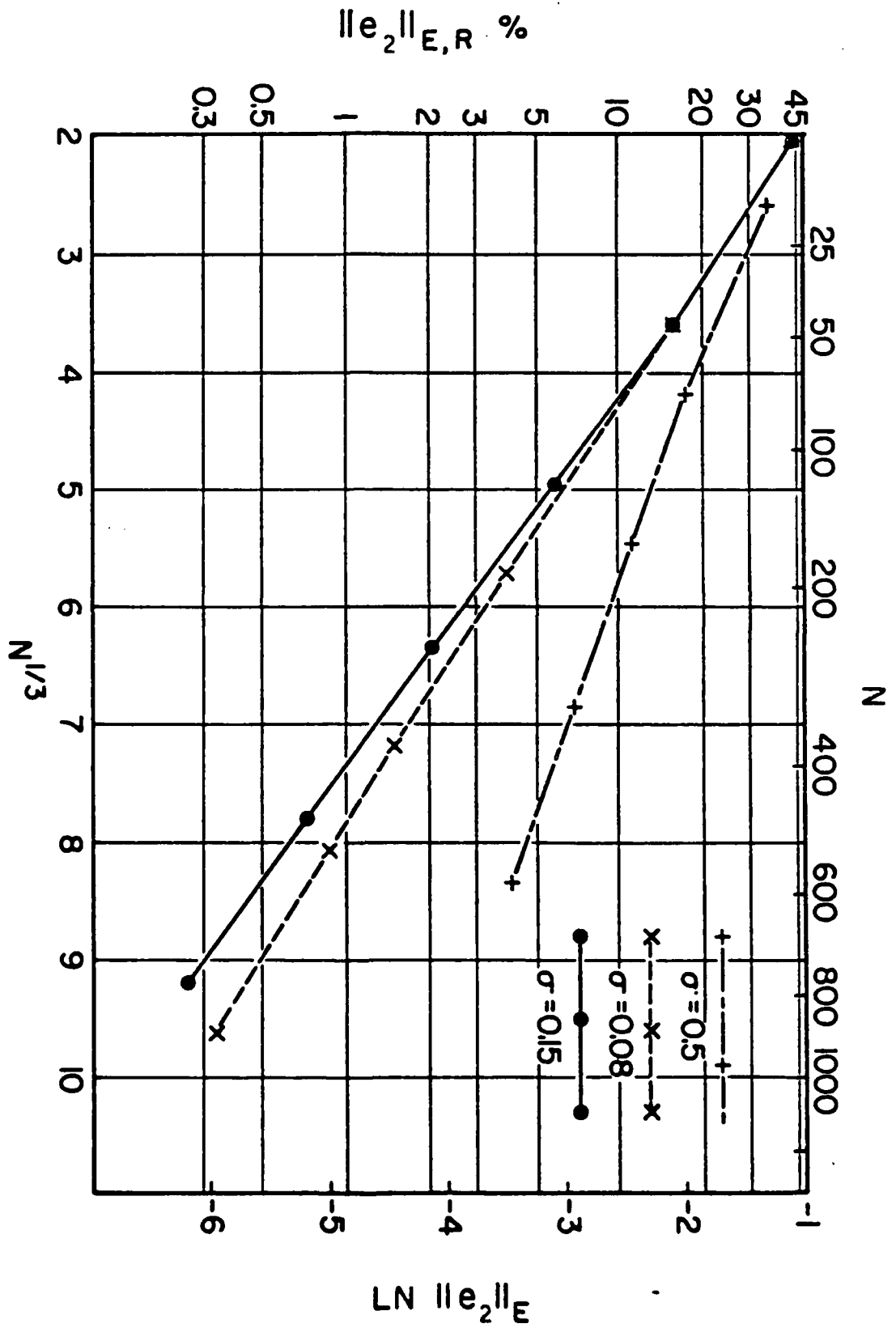


Figure 7.6

Relative Error in Energy Norm vs. Degrees of Freedom for the h-p Version  
 Problem 2 ( $E=1, \nu=.3$ ) on Mesh  $A_n, 1 \leq n \leq 6$



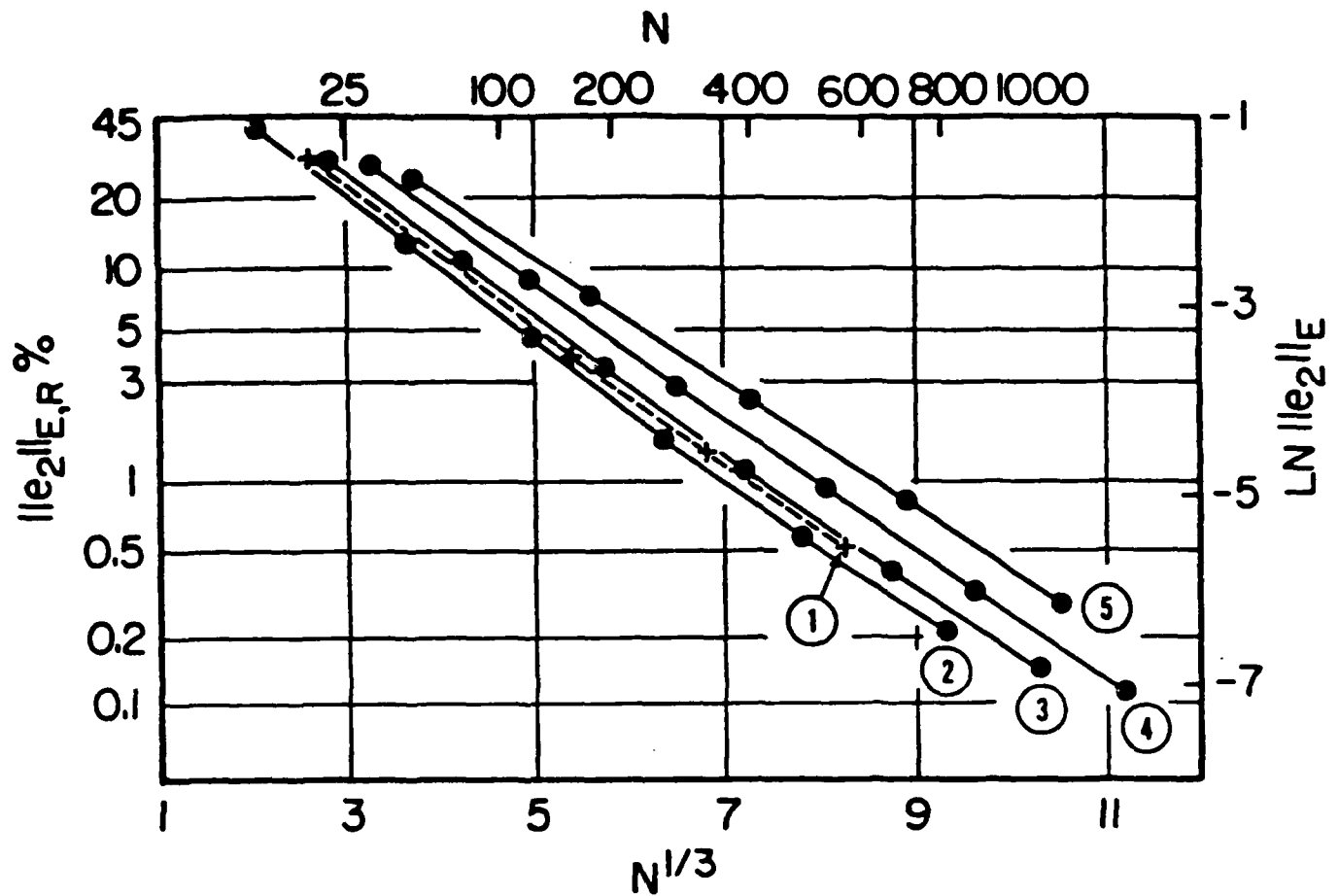


Figure 7.7

Relative Error in Energy Norm vs. Degrees of Freedom

for the h-p Version

Problem 2 ( $E=1$ ,  $\nu=.3$ ) on Mesh  $A_n$ ,  $1 \leq n \leq 6$ ,  $\sigma=.15$

- |                             |                               |
|-----------------------------|-------------------------------|
| 1 $p=n-1$ , $\mu=.8-1.0$ ,  | 2 $p=n$ , $\mu=1$ ,           |
| 4 $p=n+1$ , $\mu=1.0-1.2$ , | 3 $p=n+2$ , $\mu=1.2, -1.4$ , |
| 5 $p=n+3$ , $\mu=1.4-1.6$   |                               |

Table 7.3 gives the values of  $b_i$  and  $C_i$ . We see that the value  $p = 1$  is the best one and the sensitivity of the error on  $\mu$  is not large. The results for Problem 1 and Problem 2 are very similar. Problem 1 has been computed with meshes A and C. To achieve the same accuracy, mesh C required more computer time than mesh A.

It has been observed and rigorously analyzed in [8],[22] that the p-version is insensitive to Poisson ratio  $\nu$ . Numerical results show that the h-p version is insensitive to change of Poisson ratio  $\nu$ . In Fig. 7.8 the curve for  $\nu = 0.49$  is almost parallel to that for  $\nu = 0.3$ .

The comparison of any codes is a very delicate question because of the aims of computation, the reliability of the computed results, the ratio between the human and computer cost in the project, etc.

We address here crudely the cost of achieving the same accuracy by the h-version (elements of degree 1) with nearly optimal mesh and h-p version with nearly optimal mesh and degrees of elements. The comparison is based on the computation of Problem 1 program FEARS and PROBE (see Fig. 7.9). The error of the h-version with the optimal mesh is asymptotically ( $p = 1$ )

$$\|e\|_E \approx C_2 N^{-1/2},$$

and for the h-p version we have

$$\|e\|_E \approx C_1 e^{-b_1 N^{1/3}}.$$

The computer cost for the h-version is roughly

$$W = C + DN^{\alpha_h}$$

TABLE 7.3

Relationship between Coefficients  $b_i, c_i$  ( $i=1,2$ ) and  $\mu$   
 Problem 1 and 2 ( $E=1, \nu=.3$ ) on Mesh  $A_n, \sigma=.15, 1 \leq n \leq 6$

$\mu$	1	0.8~1.0	1.0~1.2	1.2~1.4	1.4~1.6
$p$	$n$	$n-1$	$n+1$	$n+2$	$n+3$
$b_1$	.670	.718	.680	.674	.667
$c_1 / \ u_0^1\ _E$	1.689	2.42	2.330	3.281	4.778
$b_2$	.668	.654	.683	.680	.664
$c_2 / \ u_0^2\ _E$	1.306	1.206	1.917	2.734	3.563

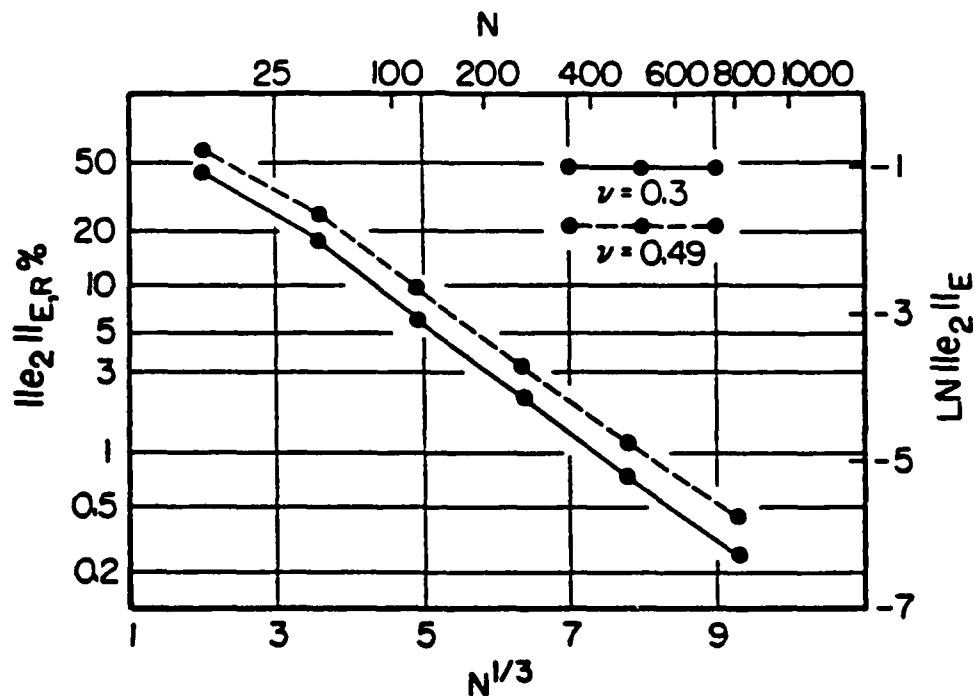


Figure 7.8

Relative Error in Energy Norm vs. Degrees of Freedom  
 for the h-p Version

Problem 2 ( $E=1, \nu=.3, .49$ ) on Mesh  $A_n, 1 \leq n \leq 6, \sigma=.15$

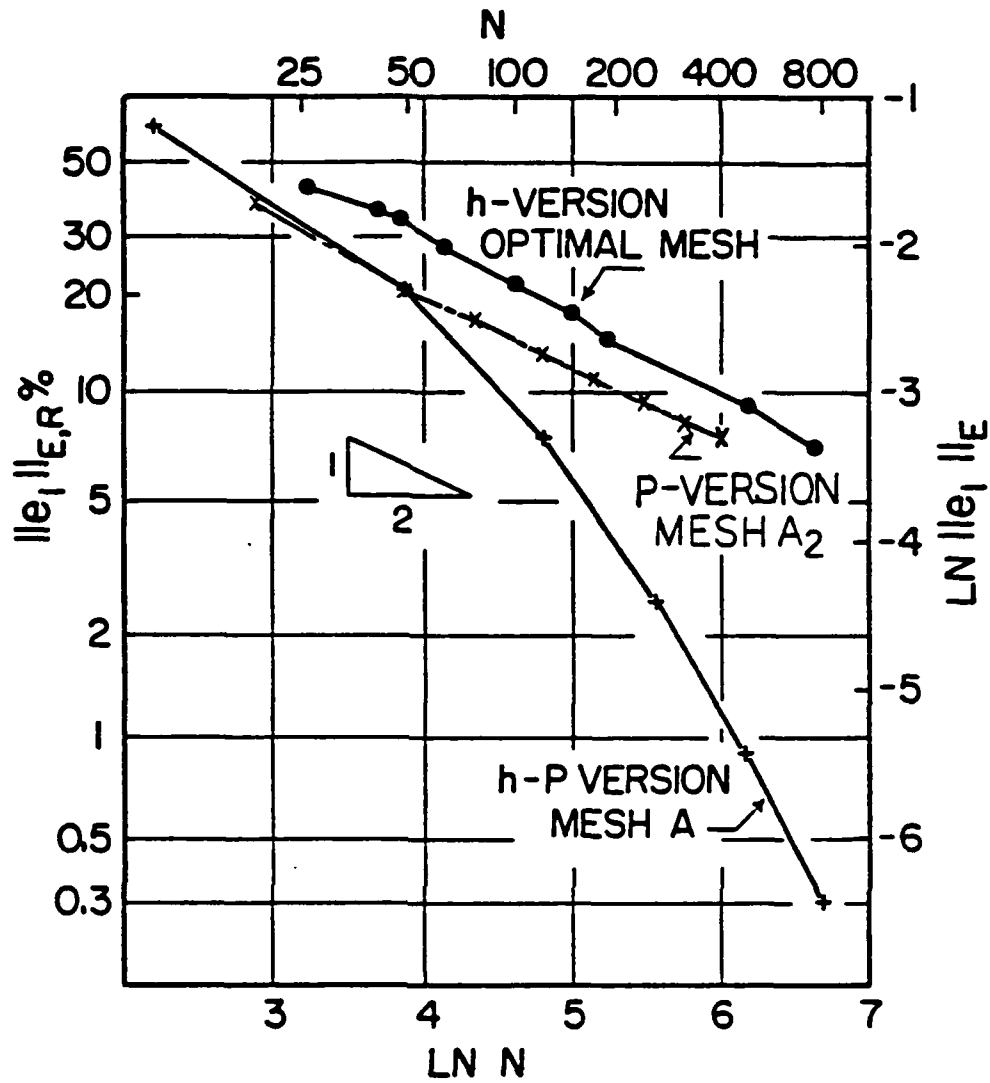


Figure 7.9

Relative Error in Energy Norm vs. Degrees of Freedom  
for the h, p, h-p Version

Problem 1 ( $E=1, \nu=.3$ ) with Mesh Factor  $\sigma=.15$

with  $\alpha_h = 1.5 \sim 2$ . FEARS uses elimination method and  $\alpha_h \approx 1.9$ . For the h-p version in our case the total number of arithmetic operations is asymptotically

$$W = C + DN^{7/3}$$

when  $N^{1/3}$  independent matrices (condensation) of order  $p^2 = O(N^{2/3})$  are decomposed and additionally a band matrix of size  $N^{2/3}$  and band  $O(N^{1/3})$  is decomposed. PROBE uses the front solver. The cost of computation of the microstiffness matrices is also  $O(N^{7/3})$ . The main cost is in the data management.

Assume that effectively

$$W \approx N^{\alpha_h}$$

for the h-version with optimal mesh and

$$W \approx N^{\beta_{h,p}}$$

for the h-p version with optimal mesh. We can compute

$$\psi(\varepsilon) = \frac{\ln N_h(\varepsilon)}{\ln N_{h,p}(\varepsilon)}$$

which is the ratio between  $\alpha_h$  and  $\beta_{h,p}$  leading to the same accuracy for given relative error  $\varepsilon$ . By  $N_h(\varepsilon)$  (resp.  $N_{h,p}(\varepsilon)$ ) we denote the number of degrees of freedom for achieving the desired accuracy. Table 7.4 shows the accuracy  $\varepsilon$ ,  $N_h(\varepsilon)$ ,  $N_{h,p}(\varepsilon)$  and  $\psi(\varepsilon)$ .

The hierarchic structure of the elements leads to hierarchic elimination so that the computation of lower degree elements (for fixed mesh) is obtainable by  $O(N)$  operation.

TABLE 7.4

Comparison of the Number of Degrees of Freedom for  $\|e_1\|_{E,R} = \epsilon\%$   
 between the h-Version with Optimal Mesh, and the h-p Version with  
 Mesh  $A_n$ ,  $1 \leq n \leq 6, \sigma = .15, \mu = 1$  on Problem 1 ( $E=1, \nu=.3$ )

$\epsilon$	30	20	10	5	3	1	0.5	0.33	0.1
$N_h(\epsilon)$	56	118	446	1397	3,883	34,945	139,782	320,892	3,494,513
$N_{h,p}(\epsilon)$	29	48	96	165	232	450	657	808	1,367
$\psi(\epsilon)$	1.195	1.232	1.337	1.418	1.517	1.712	1.826	1.894	2.087

TABLE 7.5

Estimated Error of the h-p Version

Problem 1 ( $E=1, \nu=.3$ ) on Mesh  $A_n, 1 \leq n \leq 6, \sigma = .15, \mu = 1$

$n$	$\ \tilde{e}\ _E$	$\ e\ _E$	$\ \tilde{e}\ _{E,R} \%$	$\ e\ _{E,R} \%$	$(\ e\ _E - \ \tilde{e}\ _E) / \ e\ _E \%$
1	2.9596E-1	2.9662E-1	60.83	60.92	.2189
2	9.8774E-2	9.8511E-2	20.28	20.23	-.2669
3	3.7033E-2	3.7055E-2	7.606	7.611	.0606
4	1.2489E-2	1.2500E-2	2.565	2.567	.0926
5	*4.3359E-3	4.3691E-3	*.891	.897	.6689

\* For  $n=5$  the ratio  $\Delta(4)/\Delta(5)$  is used in formula (7.13) and (7.14) instead of  $\Delta(5)/\Delta(6)$ .

Table 7.4 shows that 1% accuracy is very expensive to get by the h-version with the elements of degree 1. The accuracy 0.5% is probably not achievable at all. The h-p version allows us to use a relatively very small number of elements to obtain high accuracy.

The mesh design, although critical, is not too dependent on the geometry provided one is refining the mesh geometrically with  $\sigma = .15$  around every singular point.

In practice the true solution  $U_0$  is unknown, but error measured in the energy norm can be estimated from the energy norm of the finite element solution. Let  $E_0 = \|U_0\|_E^2$  and  $E(n) = \|U_{FE}\|_E^2$  on Mesh  $A_n$ ,  $1 \leq n \leq 6$ . By (7.1) we have for any  $\mu > 0$  asymptotically

$$E(n) = E_0 - Ce^{-bn}.$$

Therefore

$$\Delta(n) = E(n+1) - E(n) = Ce^{-bn}(1-e^{-b}),$$

$$b = \ln \frac{\Delta(n)}{\Delta(n+1)},$$

$$C = \Delta(n) \cdot \left( \frac{\Delta(n)}{\Delta(n+1)} \right)^n / \left( 1 - \frac{\Delta(n+1)}{\Delta(n)} \right).$$

$$(7.10) \quad \|\tilde{e}\|_E^2 = \frac{\Delta(n)}{\left( 1 - \frac{\Delta(n+1)}{\Delta(n)} \right)}$$

is the a-posteriori estimation of the accuracy. Further

$$(7.11) \quad \|\tilde{e}\|_{E,R} = \frac{\|\tilde{e}\|_E}{(E(n) + \|\tilde{e}\|_E^2)^{1/2}} = \frac{1}{\left( 1 + \frac{E(n)}{\|\tilde{e}\|_E^2} \right)^{1/2}} = \frac{1}{\left( 1 + \frac{E(n)}{\Delta(n)} \left( 1 - \frac{\Delta(n+1)}{\Delta(n)} \right) \right)^{1/2}}.$$

Table 7.5 shows that the true error in energy norm is estimated by  $\|\tilde{e}\|_E$  with  $\frac{\|e\|_E - \|\tilde{e}\|_E}{\|e\|_E} \leq 1\%$ . Such high reliability of the error estimation is achieved for every  $n$  because the  $h$ - $p$  version has exponential rate of convergence for very low  $n$  and  $p$  and stably maintains this behavior through all  $n$ .

Although the above numerical results are in no way exhaustive, they do suggest the following:

(1) The exponential rate of convergence of the  $h$ - $p$  version agrees in practical accuracy range with that predicted by the asymptotic approximation theory. The asymptotic rate can be achieved by low  $p$  and  $n$ , e.g.  $n = p = 1, 2$ .

(2) The coefficient  $b$  is related to singularity as well as the factor  $\sigma$  and  $\mu$ . It seems that  $\sigma \approx 0.15$  is an optimal one. Optimal  $\mu$  for uniform  $p$  depends on  $\sigma$ ,  $\beta$ ,  $d$  mesh as well as  $N$ . The computation on mesh  $A_n$ ,  $\sigma = .08$ ,  $1 \leq n \leq 6$  and mesh  $C_n$ ,  $\sigma = .15$ ,  $1 \leq n \leq 6$  shows that optimal  $\mu$  increases with  $N$ . If the asymptotic value  $\mu_0$  of the optimal  $\mu$  is known, one should select a value for  $\mu$  less than  $\mu_0$ .

3) As the  $p$ -version, the  $h$ - $p$  version is insensitive to change the Poisson ratio in plane elasticity equation. For  $\nu$  near 0.5, the exponential rate of convergence can be achieved for some low  $n$  and  $p$ . The curve of the  $h$ - $p$  version is shifted to the right without affecting the coefficient  $b$ ;

4) Since the exponential rate is achieved for low  $n$  and  $p$ , the a-posterior error estimate coincides with the true error very well from low  $p$  and  $n$  to high  $p$  and  $n$ .

5) If a higher accuracy is required then only  $h$ - $p$  version is a practical method.



## REFERENCES

- [1] Babuška, I., Aziz, A. K.: On the angle condition in the finite element method. SIAM J. Numer. Anal. 13, (1976), 214-226.
- [2] Babuška, I., Aziz, A. K.: Survey lectures on the Mathematical foundations of the finite element method. The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations. Edited by A. K. Aziz, New York: Academic Press, 1972, 3-359.
- [3] Babuška, I., Dorr, M. R.: Error estimates for the combined  $h$  and  $p$  version of finite element method. Numer. Math. 37, (1981), 252-277.
- [4] Babuška, I., Gui, W., Guo, B., Szabo, B.: Theory and Performance of the  $h$ - $p$  Versions of the Finite Element Method. To appear.
- [5] Babuška, I., Kellogg, R. B., Pitkaranta, J.: Direct and inverse error estimates for finite elements with mesh refinement. Numer. Math. 33, (1979), 447-471.
- [6] Babuška, I., Suri, M.: The optimal convergence rate of the  $p$ -version of the finite element method. To appear.
- [7] Babuška, I., Szabo, B. A., Katz, I. N.: The  $p$ -version of finite element method. SIAM J. Numer. Anal., 18, (1981), 515-545.
- [8] Babuška, I., Szabo, B. A.: On the rate of convergence of finite element method. Internat. J. Numer. Method Engrg. 18, (1982), 323-341.
- [9] Babuška, I., Szabo, B. A.: Lecture notes on finite element analysis. In preparation.
- [10] Bergh, I., Lofstrom, J.: Interpolation spaces. Berlin-Heidelberg-New York: Springer-Verlag, 1976.
- [11] Ciallet, P. G.: The finite element method for elliptic problems. Amsterdam: North-Holland, 1978.
- [12] Dorr, M. R.: The approximation theory for the  $p$ -version of the finite element method. SIAM J. Numer. Anal. 21, (1984), 1180-1207.
- [13] Dorr, M. R.: The approximation theory for the  $p$ -version of the finite element method II, SIAM J. Numer. Anal. To appear.
- [14] Gignac, D. A., Babuška, I., Mesztenvi, C.: An introduction to the FEARS program. David W. Taylor Naval Ship Research and Development Center Report DTNSRDC/CMLD-83/04, 1983.
- [15] Gelfand, I. M., Shilov, G. E.: Generalized functions. Vol. II. New York, Academic Press, 1964.

- [16] Gui, W., Babuška, I.: The h, p and h-p versions of the Finite Element Method for One Dimensional Problem, Part 1: The Error Analysis of the p-Version. Tech. Note BN-1036; Part 2: The Error Analysis of the h and h-p Versions. Tech. Note BN-1037; Part 3: The Adaptive h-p Version. Tech. Note BN-1038, IPST, University of Maryland, College Park, 1985.
- [17] Guo, B., Babuška, I.: Regularity of the solution of elliptic equations with piecewise analytic data. To appear.
- [18] Gradshteyn, I. S., Ryzhik, I. M.: Table of integrals series and products. London & New York: Academic Press, 1965.
- [19] Mesztenyi, C., Szymczak, W.: FEARS users' manual for UNIVAC 1100. Tech. Note BN-991, IPST, University of Maryland, College Park, 1982.
- [20] Muskhelishvili, N. I.: Some basic problems of the mathematical theory of elasticity (P. Noodhoff, Gronigen, Netherlands, 1963).
- [21] Szabo, B. A., Myers, K. W.: PROBE users' manual. NOETIC TECHNOLOGIES, Inc., St. Louis, MO, Nov., 1984.
- [22] Vogelius, M.: An analysis of the p-version of the finite element method for near incompressible materials, uniformly valid optimal error estimates. Numer. Math., 41, (1983), 39-53.

The Laboratory for Numerical analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.
- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.
- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.
- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Bureau of Standards.
- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.)

Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

**END**

**FILMED**

**9-85**

**DTIC**